JUNIOR HIGH SCHOOL MATHEMATICS

VOSBURGH AND GENTLEMAN

THIRD COURSE

JUNIOR HIGH SCHOOL MATHEMATICS

THIRD COURSE

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PREFACE

This book has been planned to meet the needs of the first year mathematics in the ordinary high school, as well as to serve as a Third Course in Junior High School Mathematics.

Comparison with the traditional freshman course in ordinary high schools, will show that certain geometric matter of admitted value has been inserted, and some relatively useless topics have been omitted from the algebraic portions. This renders the book particularly suitable for use as a freshman text in mathematics in ordinary high schools, and it does not detract from its value as a Third Course in Junior High School Mathematics.

The review of arithmetic and of elementary geometric and algebraic notions, with which the book begins, is very desirable for any course of this type; and it makes the work usable either with or without the preceding books in the series.

The authors have been guided in their work by the following principles:

1. That there should be a high degree of continuity in the subject matter of mathematics and in the methods of presenting it during the three years of the Junior High School.

2. That by the completion of Courses I and II, the pupil has acquired the following ideas and habits:

(*a*) By checking, the habit of assuming responsibility for the correctness of his results.

(*b*) By estimating his results in advance of the computation, a rational idea of number values.

(*c*) By the systematic and concise methods of handling the equation, an appreciation of its value as a mathematical tool in the solution of problems.

(*d*) By the use of the compasses, protractor, and ruler in geometric constructions and scale drawings, a familiarity with the properties and relations of the more common geometric figures and solids.

(*e*) By the graphic interpretation and representation of number data and of equations, an appreciation of the value of the graph in science and industry.

3. That the course in mathematics should bring the pupil who leaves school during, or at the end of, his ninth school year, in contact with adult activities that lend themselves to mathematical interpretation; and it should afford him an opportunity for the exercise of his mathematical powers through the handling of a variety of mathematical tools used in the solution of problems of everyday life.

4. That the course should aid the pupil who continues in school, in deciding whether or not he is capable of continuing his work in mathematics with profit; and it should aid him in acquiring a keener interest in the further study of mathematics. He should get from the course a clear idea of the meaning of mathematics and a vision of its manifold applications to the world's important work.

The attention of teachers is directed to the following features of this course:

Part I, Algebra.

1. The emphasis upon the idea that *per cent* is a ratio *per hundred;* the introduction of the term *per cent error* with its interesting applications; the use of the percentage formula as an application of the simple equation.

2. The continued emphasis placed upon rational methods of locating the decimal point in multiplication, division, and square root.

3. The determination of the reliability of numerical results when computed from data obtained by measurement and the number of significant figures to be retained in such results.

4. The introduction of such metric measures as have come to be generally used and whose equivalent values the pupil, as a result of his reading, should want to be able to determine.

5. The construction of the formula as a shorthand mathematical sentence and its appreciation as such when its subject is changed.

6. The simple and relatively late introduction of the negative number.

7. The simple direct presentation of the four fundamental processes with algebraic expressions.

8. The special emphasis on those forms in factoring which are applicable in the solution of equations of the second degree.

9. The practical use made of tables of square roots in evaluating irrational roots of quadratic equations.

10. The presentation of the idea of a *function* as developing from the idea of a *ratio,* which has been kept in the foreground throughout the course, and *variation* as commonly known in the quantities of industry and science.

11. The use of the graph in interpreting functions of the first and second degree.

12. The number and variety of the problems presented in the several chapters. The problems have been selected with the idea that by means of them the pupil shall develop the ability to apply general principles to new situations, shall become proficient in the use of a variety of mathematical tools, and shall acquire an appreciation of the quantitative phases of his environment.

Part II, Geometry.

1. The organized presentation and summary of geometric facts and constructions previously studied.

2. The reliance upon drawing and measurement of figures in the first geometric proofs.

3. The inductive presentation of all proofs.

4. The introduction and use of the *tangent ratio* with the use of the table of natural values to *three* figures.

5. The natural coördination of arithmetic, algebra, and geometry in the numerous applications presented.

6. The emphasis placed upon the "shop" methods in geometric constructions.

7. The considerable amount of formal geometry presented through the wise selection of major theorems.

William Ledley Vosburgh.
Frederick William Gentleman.

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JUNIOR HIGH SCHOOL MATHEMATICS

THIRD COURSE

PART I. ALGEBRA

CHAPTER I

PERCENTAGES

§ **1. Ratios.** A ***ratio*** is the expression of the quotient of two quantities. The quantities must be alike in kind and their quotients may be expressed as integers, or as fractions (common or decimal).

EXERCISES

A. Express the ratio of each of the following pairs of quantities, (*a*) in the order given, (*b*) in the reverse order.

For example, (*a*) the ratio of 3 in. to 1 ft. is $\frac{3}{12}$, or $\frac{1}{4}$, (*b*) the ratio of 1 ft. to 3 in. is $\frac{12}{3}$, or 4.

1. 4 in. to 1 ft.
2. 18 in. to 1 yd.
3. 27 in. to 1 yd.
4. 2.4 in. to 1 ft.
5. 440 yd. to 1 mi.
6. 3.6 in. to 1 yd.
7. 220 yd. to 1 mi.
8. 100 yd. to 1000 ft.
9. 4 oz. to 2 lb.
10. 12 oz. to 1 lb.
11. 500 lb. to 1 T.
12. 750 lb. to 1 T.
13. 144 cu. in. to 1 cu. ft.
14. 1.5 sq. ft. to 1 sq. yd.
15. 5.4 cu. ft. to 1 cu. yd.
16. 2 lb. 8 oz. to 15 lb.
17. 24.5 ft. to 100 ft.
18. $9\frac{3}{5}$ sec. to 1 min.
19. $15\frac{3}{5}$ sec. to 1 min.
20. 39.37 in. to 1 yd.

B. Express the ratio of each of the following pairs of numbers, (*a*) as common fractions in lowest terms, (*b*) as decimals.

For example, (*a*) the ratio of $1\frac{1}{4}$ to 2 is $\frac{\frac{5}{4}}{2}=\frac{5}{8}$, (*b*) this ratio may also be written decimally, as 0.625.

1. $2\frac{1}{2}$ to 4
2. $1\frac{3}{4}$ to 5
3. $2\frac{1}{2}$ to $7\frac{1}{2}$
4. $1\frac{1}{4}$ to $2\frac{1}{2}$
5. $10\frac{1}{2}$ to $1\frac{3}{4}$
6. $4\frac{5}{16}$ to $2\frac{7}{8}$
7. $4\frac{3}{16}$ to $8\frac{3}{8}$
8. $15\frac{3}{5}$ to 60
9. 12.5 to $9\frac{3}{4}$
10. $16\frac{13}{32}$ to $12\frac{1}{2}$
11. 25.5 to 375
12. $12\frac{3}{4}$ to $5\frac{5}{8}$
13. 75.75 to 187.5
14. 84.15 to 78.75
15. 0.125 to 2.25
16. 3.375 to 5.625
17. 0.075 to 1.2
18. 0.025 to 0.75
19. 0.005 to 0.0002
20. 0.00375 to 0.00625

§ 2. **Ratios as Per Cents.** For purposes of comparison, ratios are often expressed as per cents, that is, with the denominator 100.

EXERCISES

Express the ratio of each of the following pairs of quantities as per cents.

For example, the ratio of 3 in. to 1 ft. may be expressed as $\frac{3}{12}$, or $\frac{1}{4}$. This may be expressed also with a denominator 100, that is, as $\frac{25}{100}$. Using the term *per cent*, we may say that 3 in. is 25% of 1 ft. Since the ratio of 1 ft. to 3 in. is $\frac{4}{1}$, we may also state that 1 ft. is 400% of 3 in.

1. 8 in. to 1 ft.
2. 9 in. to 1 yd.
3. 7.2 in. to 1 yd.
4. 880 yd. to 1 mi.

5. 800 lb. to 1 T.
6. 1200 lb. to 1 T.
7. 36 sq. in. to 1 sq. ft.
8. 108 sq. in. to 1 sq. ft.
9. $2\frac{1}{4}$ sq. ft. to 1 sq. yd.
10. $6\frac{3}{4}$ sq. ft. to 1 sq. yd.
11. 576 cu. in. to 1 cu. ft.
12. 16.2 cu. ft. to 1 cu. yd.
13. 4500 lb. to 1 T.
14. 300 sq. in. to 1 sq. ft.
15. 120 in. to 2 yd.
16. 18 qt. to 3 gal.

§ 3. Per Cent Error. In the arithmetic of science and industry we reason and compute with the estimated or measured values of various quantities. As neither our estimates nor our measurements of such quantities can ever be exact, the numbers used to express them cannot be exact. For the purpose of comparison, a common way of expressing the error in such numbers is by means of what is known as the *per cent error*.

The ***per cent error*** in any quantity is the ratio of the amount of error in it to the true value of the quantity, expressed as a *per cent*. This will be made clear in the examples which follow.

EXAMPLE 1. It was estimated in advance that 45 desks would be needed in a certain classroom. It was later found that 40 desks were sufficient. What was the amount of error in the estimate? Express this error as a common fraction; as a *per cent*.

SOLUTION. 45 (Estimated number)
40 (Actual number, or *true value*)
5 (Amount of error)

$\frac{5}{40}=\frac{1}{8}$ (Error expressed as a common fraction)

$\frac{1}{8}=12\frac{1}{2}\%$ (Per cent error)

CHECK. $12\frac{1}{2}\%$ of $40=5$

$40+5=45$

Ans. 5 desks; $\frac{1}{8}$; $12\frac{1}{2}\%$.

EXAMPLE 2. The estimated cost of laying a sidewalk was \$62.50, whereas the actual cost was \$75. What was the amount of error? Express the error as a common fraction; as a *per cent.*

SOLUTION.
\$75.00 (Actual cost, or *true value*)
62.50 (Estimated cost)
\$12.50 (Amount of error)

$\frac{12.50}{75.00}=\frac{1}{6}$ (Error expressed as a common fraction)

$\frac{1}{6}=16\frac{2}{3}\%$ (Per cent error)

CHECK. $16\frac{2}{3}\%$ of $75=12.50$
$75-12.50=62.50$ *Ans.* \$12.50; $\frac{1}{6}$; $16\frac{2}{3}\%$.

EXAMPLE 3. A pupil measures the length of a line and reports it as 4.47 in. By a more careful measurement, it is found to be 4.53 in. Express the error as a common fraction; as a *per cent.*

SOLUTION.
4.53 in. (Considered as the *true value*)
4.47 in. (Pupil's measurement)
.06 in. (Amount of error)

$\frac{.06}{4.53}=\frac{6}{453}=\frac{2}{151}$ (Fractional part)

$\frac{2}{151}=1.3\%$ (Per cent error)

$$\begin{array}{r}
.013^{+}\\
151\overline{)2.000}\\
1\ 51\\
\hline
490\\
453\\
\hline
37
\end{array}$$

CHECK. 1.3% of $4.53=0.059$
$4.53-0.06=4.47$ *Ans.* $\frac{2}{151}$; 1.3%.

PROBLEMS

1. The advance estimate of the registration in a certain school was 525 pupils. The actual registration was 500 pupils. Express the amount of error in the estimate as a common fraction; as a per cent.

2. The estimated number of books needed for a certain class was 85. The actual number needed was 80. Express the amount of error in the estimate as a common fraction; as a per cent.

3. The length of a rug is reported as 7 yd. By a more careful measurement the length is found to be 6.6 yd. What is the per cent error to within a tenth of one per cent?

4. The length of the rug in Prob. 3 is reported to be 20 ft. By a more careful measurement it is found to be 19.8 ft. Express the error in per cent, to within a tenth of one per cent.

5. The estimated cost of food for a club of 10 boys for a week was $28.50. The actual cost of the food was $32. Express the error of this estimate as a per cent, to within a tenth of one per cent.

6. In a certain retail store, a weight known to be one pound is registered on one of the scales as 16.2 oz. What is the per cent error in all weights registered on this scale? Who is the loser, the retailer or the customer?

7. A yardstick is found to be 35.8 in. long. What is the per cent error in its length, to within a tenth of one per cent?

8. In checking up a record made in a race of 100 yd., it was found that the course was too short by 9 in. Express this error in per cent.

9. When the length of a quarter-mile running track was carefully measured, it was found to be too long by 2 yd. Express the error in its length in per cent, to within a tenth of one per cent.

10. The length of a meter in inches, expressed to four figures, is 39.37. When the value is taken as 39.4, what is the per cent error, to within a hundredth of one per cent?

11. A bushel is commonly regarded as containing $1\frac{1}{4}$ cu. ft.; a more accurate value is 1.244 cu. ft. When the former value is used, (*a*) what is the per cent error? (*b*) what is the amount of error in 400 bu.?

12. A cubic foot is commonly said to contain 7.5 gal.; a more accurate statement is 7.48 gal. When the former value is used, what is the per cent error, to within a hundredth of one per cent?

13. The value of π (*pi*) that you have been using is 3.14; a more accurate value is 3.1416. When the former value is used, what is the per cent error, to within a hundredth of one per cent?

14. Several pupils attempt to measure the length of a certain line correct to .01 in. A reports 4.50 in., while B reports 4.58 in. The average of all lengths reported is found to be 4.56 in. If we consider the average length as the true value, determine the per cent error in A's measurement; in B's measurement.

15. A reports the length of a desk as 2 ft. 3.8 in.; B reports it as 2 ft. 3.5 in. The average of all the separate measurements made by individuals in this class was 27.6 in. Assuming this to be the true value, determine the per cent error in A's measurement; in B's measurement.

§ 4. The Percentage Formula. All computations of percentage may be expressed by the statement,

$$\textbf{percentage} = \textbf{rate} \times \textbf{base}.$$

The formula for this statement is:

$$p = r \times b,$$

where

b = the *base* (the number on which the percentage is found),
r = the *rate* (the number of hundredths to be taken),
p = the *percentage* (the number found by taking a certain per cent of the base).

When the base is not known, the percentage formula gives us a direct way of finding it. This is shown in the solution of the examples that follow.

EXAMPLE 1. 240 is 75% of what number?

SOLUTION. Let b (base) = the number, then,

①	$240 = .75b$	
②	$240 = \frac{3}{4}b$	① ≡
③	$960 = 3b$	② × 4
④	$320 = b$	③ ÷ 3

CHECK. Substitute, in the statement of the given example, 320 for the required number. *Ans.* 320.

NOTE. The symbol, ① ≡, means that the members of equations ① and ② have identical values.

The symbol, ② × 4, means that each member of the equation ② is multiplied by 4.

The symbol, ③ ÷ 3, means that each member of equation ③ is divided by 3.

EXAMPLE 2. The total registration in a certain college in October, 1917, was 374. This was 32% below the registration in October, 1916. What was the registration in October, 1916?

SOLUTION. Let b = the number registered in 1916, then

$$1.00b - .32b = .68b \text{ (Number registered in 1917)}$$

then,

①	$.68b = 374$	
②	$b = 550$	① $\div .68$

CHECK. Substitute 550 for the registration in 1916.

Ans. 550.

EXAMPLE 3. Suits which cost $15 are to be marked so as to gain 25% on the selling price. Determine the selling price.

SOLUTION. Let c (cost) represent the percentage, and s (selling price) represent the base in the formula,

$p = r \times b$, which then becomes
$c = r \times s$.

As 25% of s is to be the profit,

75% of s must be the cost.

Hence the equation is:

①	$15 = .75s$	
②	$20 = s$	① $\div .75$

CHECK. Substitute $20 for the selling price.

Ans. $20.

PROBLEMS

[In checking, substitute *in the statement of the given problem* the result obtained.]

1. A has been able to save, during the past year, 15% of his income. If he saved $420, what was his income?

2. B has discovered that his living expenses in 1916 were $1567.80. This amount was 67% of his income. What was his income?

3. The receipts of an athletic association, from the sale of tickets for the year, amounted to $774.90. This amount was 82% of the total expenses for the year. What were the total expenses for the year?

4. In winning the pennant in the National League, in 1917, the Giants won 98 games, or 63.6% of the total number they played. What was the total number of games played by the Giants?

5. In winning the pennant in the American League, in 1917, the White Sox won 100 games, or 64.9% of the total number they played. What was the total number of games played by the White Sox?

6. The Pittsburgh team won 33.1% of the games it played in 1917. It won 51 games. How many did it play?

7. The Philadelphia team (American League) won 36% of its games. It won 55 games. How many did it play?

8. Baseball suits, on which a discount of $16\frac{2}{3}$% was allowed, cost $12.50, net. What was the original price of these suits?

9. An automobile was sold at the end of the season for $750. This was $14\frac{2}{7}\%$ below what it cost. What did it cost?

10. The production of oats in Algeria for 1917 was reported as 18,946,000 bu., or 144.2% of the 1916 crop. What was the number of bushels produced in 1916 (to the nearest thousand)?

11. New York State produced 877,000 bu. of beans in 1917. This was 121.8% of the number of bushels it produced in 1916. What was the production in 1916 (to the nearest thousand)?

12. The country price of a bushel of wheat on October 1, 1917, was published as $2.066, or 151.6% of the price on October 1, 1916. What was the price on October 1, 1916?

13. The country price of a bushel of corn on the same date, October 1, 1917, was $1.751. This was 212.8% of the price on October 1, 1916. What was the price on October 1, 1916?

14. At the end of the season an automobile was sold for $700. This was 24.3% below what was paid for it. What was the amount paid for it (to the nearest dollar)?

15. A dealer sold two cows at $95 each. On one he gained 25% of the cost while on the other he lost 20% of the cost. Did he gain or lose on the transaction? How much?

16. A retailer sold two suits at $30 each. On one he made a profit of 50% of the cost while on the other he lost 10% of the selling price. What was his net profit?

17. The following articles are to be marked to sell at the per cent of profit indicated on the selling price. Find the selling price in each instance.

	Article	Cost	Profit on Selling Price	Selling Price
(*a*)	Chair . . .	$2.25	25%	?
(*b*)	Table . . .	$22.50	$37\frac{1}{2}$%	?
(*c*)	Davenport .	$75.00	$62\frac{1}{2}$%	?
(*d*)	Rug . . .	$84.00	75%	?

18. A merchant marks his goods 50% above cost and gives his customers a discount of 10% of the marked price. Determine his per cent of profit: (*a*) on the cost, (*b*) on the selling price. (SUGGESTION. Let c = the cost.)

19. A merchant marks his goods $37\frac{1}{2}$% above cost and allows a discount of 8% of marked price. Determine his per cent of profit: (*a*) on the cost, (*b*) on the selling price.

20. If water expands 10% when it freezes, how much does ice contract when it turns into water?

21. In settling with his creditors, A was able to pay 92¢ on $1. One of the creditors received $529. What was the original amount of his bill?

22. After receiving an increase of 40% in wages, carpenters are getting $7.14 per day. What was their wage per day before the increase?

23. A coat which cost $12 was marked to sell at a profit of 50% on the selling price. When the coat was sold the price was reduced 25%. What was the per cent of profit realized on the price at which it was sold?

24. A merchant marks his goods to sell at wholesale at a profit of 20% on the selling price. His retail price is 20% above the wholesale price. What is his per cent of profit on the selling price when he sells at retail?

25. A table was sold on a special sale for $22.50. This was at a discount of 25% from the marked price. What was the marked price?

26. Two successive discounts of 10% and 2% on a bill of goods amount to $17.70. What was the original amount of this bill?

27. At what price must an article costing $1.20 be marked so that, after deducting 20% from the marked price, a profit of 25% may be made on the cost?

28. At what price must an article costing $1.20 be marked so that, after deducting 20% from the marked price, a profit of 25% may be made on the selling price?

29. How much is gained on one dozen articles that cost $2.50 each when they are sold so as to gain 20% on the selling price?

30. The publisher's price of a certain book is 75¢. The publisher sells the book to retailers at a discount of 20% from this price. If the retailer sells this book for 75¢, what per cent does he gain, (*a*) on the cost? (*b*) on the selling price?

CHAPTER II

APPROXIMATE COMPUTATION

§ 5. Approximate Products. When numbers that are derived by measuring are met in computation, the degree of accuracy obtainable in results depends upon the kind of quantity measured and upon the precision with which the measurements are made, — not upon the number of figures actually used in performing the computation. A result can be no more accurate than the least accurate datum.

NOTE. Commercial problems dealing with dollars and cents should not be confused with problems dealing with numbers obtained by measurement. In a commercial problem, when the number of units is known and the price per unit is given, the result should be computed to the nearest cent.

When we measure the length of a room and record it as 37.4′, we generally mean that the true length is nearer to 37.4′ than it is to 37.3′ or to 37.5′. The length might be nearly 37.45′ or slightly more than 37.35′. Similarly, we might record the width as 23.8′, meaning that it represented the width to the nearest tenth of a foot. We shall find that the area, computed from these measurements, will not be dependable beyond the nearest square foot. (See Ex. 1, which follows.)

EXAMPLE 1. The length of a room is 37.4′ and its width is 23.8′. Find the area of the floor in square feet.

FORMULA. $A = bh$

$$A = 37.4 \times 23.8$$

SPECIMEN SOLUTIONS.

A. The work appears as follows, if the customary method of multiplication is used:

37.4	
23.8	
2992	Estimate
1122	$20 \times 40 = 800$
748	
890.12	

The area of the floor appears to be 890 sq. ft. to the nearest square foot.

B. If the order of the multiplying is reversed; that is, if 37.4 is multiplied by the left-hand digit of the multiplier first, the work will appear as follows:

37.4		
23.8		
748.	(20×37.4)	Estimate
112.2	(3×37.4)	$20 \times 40 = 800$
29.92	$(.8 \times 37.4)$	
890.12		

In this method the decimal point is located in the first partial product to correspond with the estimate.

NOTE. A more complete explanation of this method of multiplying can be found in Chapter I, First Course.

C. The measurements of the floor, of which we are finding the area, were made only to the nearest tenth of a foot. It will now be shown that this area cannot be accurate beyond the third figure, and that even that figure is in doubt.

37.4	
23.8	
748.**	Estimate
112.2*	$20 \times 40 = 800$
29.92	
890.12	

The area of the floor is computed as 890.12 sq. ft. In the first partial product, however, the places marked with stars (*) are not necessarily zeros. Had the length been measured to the nearest hundredth of a foot, some figure would have appeared after 748 in the column following the decimal point. The stars show that the figures in these columns are doubtful. In fact the first partial product, 748, might be either 747 or 749; for 20 times 37.35 is 747, and 20 times 37.44 is 749 to the nearest third figure. (Any measurement between 37.35′ and 37.45′ would be called 37.4′, to the nearest tenth.)

In the final product, therefore, the figures that appear in the doubtful columns (which happen, in this example, to be at the right of the decimal point) are of little use, as their retention does not make the answer any more accurate than it would have been without them. This final product might be as small as 887.1 (37.35×23.75), or as large as 892.6 (37.44×23.84), depending upon the data. Hence the *third* figure of this answer is in *doubt* and we waste labor by keeping figures in the partial products that do not appear in our final product.

D. If we cut the partial products, by not writing any figures which would come in the doubtful columns, we shall obtain the figures necessary for the final product. The work is shown on page 16.

$3\acute{7}.\acute{4}$			
$\underline{23.8}$			
748.			Estimate
112.	($3\times37+1$.	See Note 1.)	$20\times40=800$
$\underline{30.}$	($.8\times30+6$.	See Note 2.)	
890.			*Ans.* $A=890$ sq. ft.

Note 1. ($3\times37.\acute{4}=112.2$.) As the .2, if written, would come in a doubtful column (see Solution C), we do not write it; but in multiplying 37 by 3, we notice that we have 1 to carry (since $3\times.4=1.2$, which is nearer 1 than 2). We now place a mark (′) over the 4 in the multiplicand. This is to show that we have not written a figure in the partial product that would come in a doubtful column, but that we have noted, in multiplying, its effect upon the next column to the left.

Note 2. ($.8\times3\acute{7}.\acute{4}=29.92$.) As the 9 and the 2, if written, would both come in doubtful columns (see Solution C), we do not write them; but in multiplying 30 by .8, we notice that we had 6 to carry (since $.8\times7=5.6$, which is nearer 6 than 5). We now place a mark (′) over the 7 in the multiplicand to indicate again that we have not written a figure that would come in a doubtful column; but that we have noted, in multiplying, its effect on the next column to the left.

Note 3. Measurements, like the above, made to the nearest third figure, give a result not reliable beyond three figures, hence figures beyond the third may be neglected. Even the third figure may not be reliable. If the first three figures of a result are reliable, such a result is said to be of *three-figure accuracy.*

Example 2. A table top is 8.28′ long and 3.26′ wide. How many square feet does it contain? (Find your result to tenths of a square foot.)

Formula. $A=bh$

$A=8.28\times3.26$

$A=27.0$

(The work follows.)

Specimen Solution.

```
 8.28
 3.26
24.84  (3×8.28)                              Estimate
 1.66  (.2×8.28.   See Note 1.)              3×9=27
  .49  (.06×8.2.   See Note 2.)
27.0                                  Ans. A=27.0 sq. ft.
```

Note 1. (.2×8.28.) Place mark (′) over 8; multiply 8.2 by .2 and add .02 (.2×.08=.016, which is nearer .02 than .01).

Note 2. (.06×8.2.) Place mark (′) over 2; multiply 8 by .06 and add .01 (.06×.2=.012, which is nearer .01 than .02).

Note 3. Since the result was required to tenths, it was necessary to write 0 after the decimal point in the answer.

EXERCISES

A. By the above method, in each of the following exercises, find the product to three figures.

1. 3.76×25.2
2. 8.34×3.61
3. 32.5×4.24
4. 8.46×3.14
5. 42.5×16.7
6. 8.39×6.06
7. 4.08×67.2
8. 8.05×9.03
9. 775×4.06
10. 8.70×2.06
11. 68.7×0.804
12. 4.04×20.4

B. In each of the following exercises, find the product to four figures.

1. 39.37×18.75
2. 25.72×62.44
3. 523.8×12.56
4. 45.36×61.16
5. 4.837×12.18
6. 7.854×24.34
7. 65.75×0.7854
8. 24.75×3.142
9. 40.08×3.142
10. 25.08×20.05

§ 6. **Approximate Quotients.** The ratio of the circumference of a circle to its diameter is π. This is expressed by the equation

$$\frac{c}{d}=\pi.$$

When the circumference of a circle is known, the above equation may be transformed into a useful formula for finding the diameter. The steps in the transformation are as follows:

①	$\frac{c}{d}=\pi$ (Given)	
②	$c=\pi d$	① $\times d$
③	$\frac{c}{\pi}=d$	② $\div \pi$

This formula is a brief way of saying that, when the circumference of a circle is known, the diameter may be found by dividing the circumference by π.

In performing the process of division, since the values of both π and of the circumference are known only approximately, we may contract our work as shown in the model examples which follow.

Example 1. The circumference of a circle is 30.6 in. Find its diameter to three figures.

Solution.

$$d=\frac{c}{\pi} \quad \text{(Formula)}$$

$$d=\frac{30.6}{3.14}$$

$$d=9.75$$

Estimate

$$\frac{30}{3}=10$$

The circumference is given as 30.6 in. to the nearest third figure, hence it might be nearly 30.65 in. or slightly more than 30.55 in.

Dividing 30.64 by 3.1416 in the usual way, the quotient is 9.753 (9.75 to the nearest third figure).

Dividing 30.55 by 3.1416, the quotient is 9.724 (9.72 to the nearest third figure).

Since the third figure of the quotient is in doubt, the computation will be sufficiently accurate if done as follows.

Determine the location of the decimal point in the quotient by means of the estimate.

```
 ' ''  9.746
3.14)30.6*
     28 26   (9×3.14 to four figures.   See Note 1.)
      2 34
      2 20   (7×3.14 to three figures.   See Note 2.)
        14
        12   (4×31 to two figures.   See Note 3.)
         2
         2   (6×3 to one figure.   See Note 4.)
```

NOTE 1. In the first partial product (2826), although the 6 comes in a doubtful column, it should be retained. It is advisable to keep all figures of the first partial product.

NOTE 2. (7×314 = 2198.) The 8 in the partial product, if written, would come in a column not to be retained. Do not write it, but instead carry 3 (as 7×4 = 28, which is near 30) to 7 in the next column. Place the (′) over the 4 of the divisor to show that we have noted, in the partial product, its carrying effect.

NOTE 3. (4×31 = 124.) Write 12 in the partial product and place the mark (′) over the 1 in the divisor for the reason given in Note 2.

NOTE 4. (6×3 = 18.) As 18 is near 20, place the mark (′) over the 3 in the divisor and write 2 to carry in the partial product.

CHECK. In checking, multiply the quotient by the divisor, *not* the divisor by the quotient. In multiplying, reject figures that would come in doubtful columns as shown in the work in multiplication. (See page 20.)

$$\begin{array}{r} 9.75 \\ \underline{3.14} \\ 29.25 \\ .98 \\ \underline{.39} \\ 30.62 \end{array}$$

(30.6 to three figures)

Ans. The diameter is 9.75″.

The value of π is 3.14159 ⋯. In all problems in which π is the divisor, use at least as many figures for the value of π as are given in the data of the problem. It is customary to use not less than three figures in any case.

Example 2. The circumference of a circle is 51.26 in. Find its diameter.

Solution. $d=\frac{c}{\pi}$ (Formula)

$d=\frac{51.26}{3.142}$

$d=16.32$

Estimate

$\frac{51}{3}=17$

$$\begin{array}{r} 16.316 \\ 3.142\overline{)51.26} \\ \underline{31\ 42} \\ 19\ 84 \\ \underline{18\ 85} \\ 99 \\ \underline{94} \\ 5 \\ \underline{3} \\ 2 \\ \underline{2} \end{array}$$

CHECK.

$$\begin{array}{r} 16.32 \\ \underline{3.142} \\ 48.96 \\ 1.63 \\ .65 \\ \underline{.03} \\ 51.27 \end{array}$$

Ans. $d = 16.32$ in.

EXAMPLE 3. The surface of a rectangular plate is 42.3 sq. in. The length of the plate is 8.27 in. Find its width.

SOLUTION. Transforming the formula,

1 $$A = bh$$

2 $$\frac{A}{b} = h \qquad 1 \div b$$

Substituting the values of A and b in 2.

3 $$\frac{43.2}{8.27} = h$$

Estimate

$$\frac{40}{8} = 5$$

$$h = 5.11$$

$$\begin{array}{r} 5.114 \\ 8.27\overline{)42.3*} \\ \underline{41\ 35} \\ 95 \\ \underline{83} \\ 12 \\ \underline{8} \\ 4 \end{array}$$

CHECK.

$$\begin{array}{r} 5.11 \\ \underline{8.27} \\ 40.88 \\ 1.02 \\ \underline{.36} \\ 42.26 \end{array}$$

(42.3 to three figures) *Ans.* $h = 5.11$ in.

EXERCISES

A. In each of the following exercises with circles whose circumferences are known, find the required values.

FORMULA. $d = \frac{c}{\pi}$.

	c	d	CHECK
1.	56.4″	?	?
2.	69.7″	?	?
3.	8.58′	?	?
4.	9.45′	?	?
5.	27.6″	?	?
6.	30.8″	?	?
7.	1.67′	?	?
8.	1.23′	?	?
9.	43.26″	?	?
10.	82.78″	?	?
11.	31.29″	?	?
12.	21.17″	?	?
13.	31.2′	?	?
14.	29.8′	?	?
15.	31.29′	?	?

B. In each of the following exercises with rectangles whose areas are known, find the required values.

FORMULAS. $b = \frac{A}{h}$, $h = \frac{A}{b}$.

	A	b	h	CHECK
1.	45.3 sq. ft.	18.4′	?	?
2.	49.1 sq. ft.	2.82′	?	?
3.	82.7 sq. ft.	?	24.6′	?
4.	64.2 sq. ft.	?	4.67′	?
5.	308 sq. in.	73.6″	?	?
6.	350 sq. in.	?	65.4″	?
7.	115.6 sq. in.	23.4″	?	?
8.	201.5 sq. in.	?	56.4″	?
9.	35.48 sq. ft.	?	76.2′	?
10.	41.89 sq. ft.	83.7′	?	?

§ 7. **Square Root.** The ***square root*** of a number is one of the two equal factors of that number. Hence, when a number is divided by its square root, the quotient equals the divisor. It necessarily follows that when the divisor of a number is smaller than its square root, the quotient is larger than the square root; when the divisor is larger than the square root, the quotient is smaller. In these cases, the true square root will generally be very nearly halfway between the divisor and the quotient.

The application of this principle gives us a method of finding, *by trial*, the square root of any number. A study of the examples which follow will make this clear.

EXAMPLE 1. Find the square root of 34.3 to *three* figures.

SOLUTION. Since $5^2 = 25$ and $6^2 = 36$, try a divisor near 6, say 5.8.

```
      5.92
5.8)34.3
    29 0
     5 3
     5 2
       1
       1
```

Since the quotient is 5.92 and the divisor is 5.8, try a divisor halfway between; that is, 5.86. Repeat the process of division.

```
       5.85
5.86)34.3*
     29 30
      5 00
      4 69
        31
        29
```

Evidently the square root is very nearly 5.855. Retaining *three* figures (the last one, as an even number), we have 5.86 for the square root.

This should now be checked by multiplication. (See page 24.)

CHECK.

```
 5.86
 5.86
-----
29.30
 4.69
  .35
-----
34.34
```

Ans. $\sqrt{34.3} = 5.86.$

EXAMPLE 2. Find the square root of 657.4 to *four* figures.

SOLUTION. Since $25^2 = 625$ and $26^2 = 676$, try a divisor near 26, say 25.8.

```
         25.48
  25.8)657.4
       516
       -----
       141 4
       129 0
       -----
        12 4
        10 3
       -----
         2 1
         2 1
       -----
```

We shall now try a divisor halfway between 25.48 and 25.8; that is, 25.64.

```
          25.64
  25.64)657.4
        512 8
        -----
        144 6
        128 2
        -----
         16 4
         15 4
        -----
          1 0
          1 0
        -----
```

This answer may now be checked by multiplication, as before.

Ans. $\sqrt{657.4} = 25.64.$

NOTE. To find the square root of a number to *three* figures, you will have to repeat the process of division if, in the first division, the divisor differs from the quotient by more than *one* in the second figure. To find the square root of a number to *four* figures, you will have to repeat the process of division if the first two figures of the divisor and quotient do not agree in the first division. It is advisable to check the answer in all cases by contracted multiplication.

EXAMPLE 3. Find the square root of 7 to *four* figures.

SOLUTION. Consider that 7 is an *exact* number and place three zeros after the decimal point. Since $2.5^2 = 6.25$, try 2.6 as a divisor.

1st Division

```
     2.692
2.6)7.000
     5 2
     1 80
     1 56
      240
      234
        6
        5
```

2d Division

```
        2.646
2.646)7.000
        5 292
        1 708
        1 588
          120
          106
           14
           16
```

This answer (2.646) may be checked by multiplication, or by consulting a Table of Square Roots.

Ans. $\sqrt{7} = 2.646$.

EXERCISES

A. Find the square root of each of the following numbers to *three* figures.

1. 237.	**6.** 52.4	**11.** 2.43
2. 538.	**7.** 87.6	**12.** 5.24
3. 910.	**8.** 0.910	**13.** 8.76
4. 876.	**9.** 0.876	**14.** 0.0538
5. 24.3	**10.** 0.243	**15.** 0.0524

B. Find the square root of each of the following numbers to *four* figures.

1. 24.36	6. 8985.	*11. 2.
2. 28.54	7. 9234.	12. 3.
3. 57.96	8. 7.892	13. 5.
4. 45.84	9. 0.7892	14. 8.
5. 2.854	10. 0.5642	15. 12.

*NOTE. Consider the numbers in Exs. 11–15 as *exact* numbers, and place as many zeros after the decimal point as are needed to find the required number of figures in the root.

PROBLEMS

1. The area of a square lot of land is 458 sq. ft. Find the length of one side.

2. Find the side of a square that has an area of 35.2 sq. in.

3. Find the radius of a circle whose area is 145 sq. in. (FORMULA. $r=\sqrt{\frac{A}{\pi}}$.)

4. Find the radius of a circular metal plate if its area is 41.5 sq. in.

5. Find the diameter of a water pipe if its cross sectional area is 5.30 sq. in. (*Suggestion.* Find the radius first, using the formula in Prob. 3.)

6. A rectangle is 13.4″ long and 12.3″ wide. Find the side of the square which has the same area as the rectangle.

7. The sides of a right triangle are $12\frac{1}{2}''$ and $18\frac{1}{2}''$. Find its hypotenuse. (Formula: $c=\sqrt{a^2+b^2}$, where c is the hypotenuse and a and b are the two sides.)

8. The sides of a right triangle are each 4.25″. Find its hypotenuse.

CHAPTER III

CONSTRUCTION OF FORMULAS

EXERCISES

§ 8. **Introduction.** The construction of formulas is illustrated by the mensuration formulas in Course II, by the percentage formula (§ 4), and other rules already given. The following exercises give practice in actually making new formulas.

Solve each of the following exercises and explain your solution.

1. A boy walks at the average rate of $3\frac{1}{2}$ miles an hour. How far does he go in 3 hours? In x hours?

2. A train travels at an average rate of 40 miles per hour. How far does it go in $2\frac{1}{2}$ hours? In $3x/4$ hours?

3. A steamship moves at a uniform rate of $22\frac{1}{2}$ miles an hour. How far does it go in 4 hours? In y hours?

4. A boy runs at an average rate of 6 miles an hour. How far does he go in 2 hours and 15 minutes? In $x/4$ hours?

5. An airplane is traveling at a uniform rate of 48 miles an hour. How far does it go in 45 minutes? In y minutes?

6. In each of the exercises 1–5 there are three quantities that have a certain relation to each other. The quantities are *distance*, *rate* and *time*. Using the letters d, r, and t write a formula to show their relation. Explain what each letter stands for.

7. Find the average rate of a train that goes 150 miles in 5 hours. (Use the formula obtained in Ex. 6.)

8. Find the average rate of an automobile that goes 198 miles in 9 hours.

9. Find the average rate of a bicyclist who goes 32 miles in 2 hours and 30 minutes.

10. Find the average rate of a runner (in feet per second), who goes a mile in 5 minutes and 12 seconds. Why is the *average* rate required rather than the *uniform* rate?

11. An express train travels at an average rate of 44 miles an hour, and passes two towns 22 miles apart. How long does it take the train to go from one town to the other?

12. An express train travels at an average rate of 40 miles an hour, and passes two towns 12 miles apart. How long does it take the train to go from one town to the other?

13. A freight train is traveling at an average rate of 25 miles an hour. How long will it take to go from one city to another, if the cities are 70 miles apart?

14. In going from Boston to New York, 233 miles, an express train takes 6 hours. What is its average rate per hour?

15. A train leaves Boston at 9.00 A.M. and reaches Portland, Me., at 12.10 P.M., a distance of 108 miles. What is the average rate per hour to the nearest mile?

16. A special train of cars is to take a party of men a distance of 550 miles in $10\frac{1}{2}$ hours. What must be the average rate per hour to the nearest mile?

17. Which is the greater speed, a runner going 100 yards in 9.5 seconds, or a train going 25 miles an hour?

18. I see a flash of a gun and hear the sound 5 seconds later. How far away is the gunner? (Sound travels at the rate of 1080 feet per second. Since light travels at a speed of approximately 186,000 miles per second, do you need to consider the time that it took the flash to reach you?)

19. The echo of a sound returns from a cliff in 3 seconds. How far away is the cliff?

20. A flash of lightning is seen, and the thunder peal is heard 5 seconds later. How far away did the lightning strike?

21. A flash of lightning is seen, and the thunder peal is heard 9 seconds later. How many feet away did the lightning strike? Is your result probably the correct distance to the nearest foot? Explain your answer.

22. A flash of lightning is seen, and the thunder peal is heard 4 seconds later. Find the distance to the nearest tenth of a mile. Is the accuracy required here reasonable? Why?

23. A tramping party is walking at the average rate of $3\frac{1}{2}$ miles an hour. They leave a certain town at 8 A.M. Taking out one hour for lunch and rest, they reach another town at 5 P.M. How many miles apart are the towns? Does this result show the exact distance or the approximate distance? Why?

24. What is a pedometer? (Consult a dictionary.) Explain how it works. The leader of a tramping party, after traveling for 5 hours, finds that his pedometer registered $17\frac{1}{2}$ miles. What has been the average rate of the

party per hour? What can you tell about the accuracy of the number of miles registered by the pedometer?

25. If it is necessary for you to walk to a place exactly one mile distant and to be there at a definite time, how accurately can you tell when you must start?

26. Could you find the distance between two places by walking from one to the other with a watch in hand? Explain. If the distance is over a mile how accurately do you think you could measure it in this way?

27. A street rises 20 ft. in a horizontal distance of 400 ft. What is the slope?

The grade or slope of a street tells how steep the street is. It is often expressed in per cent; that is, as a rise of a certain number of feet to the hundred.

In this problem, since the street rises 20 ft. in a horizontal distance of 400 ft., it is evident that it rises 5 ft. to the hundred. Why?

This is called a 5% grade.

28. The formula for the relation expressed in Ex. 27 is: $g=\frac{r}{b}\cdot$ (Fig. 1.) Explain.

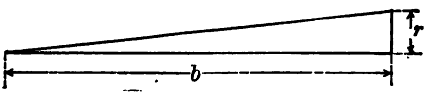

FIG. 1.

29. What is the grade of a street that rises 32 ft. in 256 ft.? Express it as a decimal and as a per cent.

30. A bridge is to be 10 ft. above the level of the road. If the grade must not be more than 20%, how long must the approach to the bridge be?

31. A street rises 43 ft. in a horizontal distance of 137 ft. Find its slope or grade, within a tenth of one per cent.

32. Balance a yardstick at its center. Place a 3-lb. weight on one part 12 in. from the center and a 4-lb. weight on the other part, so as to balance the 3-lb. weight. How far is the 4-lb. weight from the center?

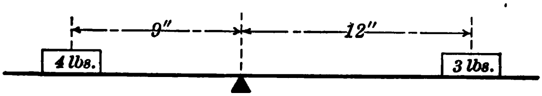

FIG. 2.

Figure 2 represents the ***lever*** described in Ex. 32. The balancing point is called the ***fulcrum*** of the lever.

The product of one weight by its distance from the fulcrum equals the product of the other weight by its distance from the fulcrum; that is, $4 \times 9 = 3 \times 12$.

33. Two boys are balanced on a teeter board. One of the boys weighs 80 lb. and is 3 ft. from the fulcrum. If the other boy weighs 60 lb., how far will he be from the fulcrum? (The equation is: $80 \times 3 = 60d$.)

34. One boy weighing 100 lb. is 4 ft. from the fulcrum of a teeter board. Another boy just balances him at a distance of 5 ft. from the fulcrum. What is the weight of the second boy?

35. A and B are 5 ft. and 7 ft. from the fulcrum of a balanced teeter board. A weighs 84 lb. What is the weight of B?

36. In each of the Exs. 32–35 there are four quantities that have a certain relation to each other. The quantities

are the two weights, w_1 and w_2; and their respective distances from the fulcrum, d_1 and d_2. Write a formula to show the relation, and explain what each letter stands for.

37. What is the weight of an object 10 in. from the fulcrum of a lever, if it balances a weight of 6 lb. at 14.5 in. from the fulcrum? (Use the formula obtained in Ex. 36.)

38. What is the weight of an object $7\frac{1}{2}$ in. from the fulcrum of a lever, if it balances a weight of $4\frac{1}{2}$ lb. at 10 in. from the fulcrum?

39. To raise a stone of 250 lb., a man places a crowbar so that the fulcrum is 6 in. from the point at which the crowbar touches the stone (Fig. 3). What force must be applied to the crowbar 24 in. from the fulcrum to raise the stone?

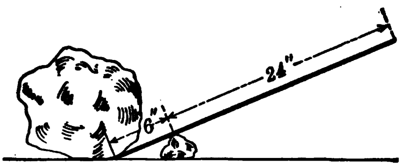

FIG. 3.

40. In Ex. 39, suppose the force is applied 3 ft. from the fulcrum, what force is necessary?

41. A pole 10 ft. long is used to lift the wheel of an automobile. One end of the pole is placed under the hub of the wheel $1\frac{1}{4}$ ft. from the fulcrum. If the wheel is carrying a load of 800 lb., what force will need to be applied at the other end of the pole to lift the wheel?

§ 9. **Mathematical Symbols.** Formulas are used to express in abbreviated form the rules of mensuration and mechanical laws. Thus long rules and tedious explanations are avoided, and the solution of problems is made simple and direct.

In order to construct a formula from a rule, the ability to do the two things that follow, is essential:

(1) To interpret correctly from the language of the rule what mathematical operations are required and in what order they are to be performed.

(2) To express accurately these operations and relations by the use of necessary mathematical symbols.

EXERCISES

Express mathematical symbols for each of the following phrases.

1. The sum of two numbers, a and b. *Ans.* $a+b$.

2. The difference of two numbers, a and b.

3. The product of two numbers, a and b.

(The product of two numbers, such as a and b, may be written $a \times b$, or $a \cdot b$, or ab. The dot ($\cdot$) is often used instead of the sign $\times$ to avoid confusion with the letter x.)

4. The quotient of two numbers, a and b.

5. The ratio of a to b.

6. The sum of three times x and twice y.

7. The difference between five times x and three times y.

8. The product of three times x and four times y.

9. The quotient of x and four times y.

10. The sum of one half a and two thirds b.

11. The ratio of 3 times x to y.

12. The square of c.

13. The sum of the squares of a and b.

14. The difference of the squares of x and y.

15. The quotient of the square of a and three times b.

16. The ratio of the square of b_1 to the square of b_2.

17. Three times the difference of c and d.

18. One half the sum of m and p.

19. The square of the sum of x and y.

20. The square of the difference of p and q.

21. The sum of the squares of a and b plus twice their product.

22. The sum of the squares of a and b minus twice their product.

23. The cube of a.

24. The sum of the cubes of a and b.

25. The ratio of the cubes of x and y.

§ 10. Formulas of Mensuration.

Example. Write a formula for the following rule and draw a figure. The area of a rectangle equals the product of its base and height. $(A, b, h.)$

Solution. The formula is

$$A = bh,$$

where

A = the area of a rectangle (in square units),

b = the base (in linear units),

h = the height (in linear units).

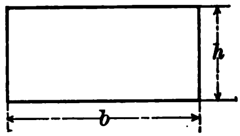

FIG. 4.

NOTE. In explaining the letters in a formula, it is necessary to name the *kind* of *unit* used, when possible.

EXERCISES

Write a formula for each of the following rules, explaining each letter used, and draw a figure.

(The letters in the parentheses suggest the letters that may well be used in each exercise. For definitions of geometric terms see Part II.)

1. The area of a triangle equals one half the product of its base and height. (A, b, h.)

2. The area of a trapezoid equals one half the product of the sum of its bases by its height. (A, b_1, b_2, h.)

3. The area of a square equals the square of one side. (A, s.)

4. The perimeter of a rectangle equals twice its base plus twice its height. (p, b, h.)

5. The perimeter of a square equals four times one side. (p, s.)

6. Circumference of a circle.

(a) The circumference of a circle equals π times its diameter. (c, d.)

(b) The circumference of a circle equals 2π times its radius. (c, r.)

(Formulas (a), in Exs. 6–7, are commonly used in science.)

7. Area of a circle.

(*a*) The area of a circle equals $\frac{1}{4}\pi$ times the square of its diameter. (A, d.) (See note at foot of page 35.)

(*b*) The area of a circle equals π times the square of its radius. (A, r.)

8. The three sides of a right triangle.

(*a*) The square of the hypotenuse of a right triangle equals the sum of the squares of its two sides. (c, a, b.)

(*b*) The square of one side of a right triangle equals the square of its hypotenuse minus the square of the other side.

9. The sum of the three angles of a triangle equals 180°. (A, B, C.)

10. The volume of a rectangular block equals the product of its length, width, and height. (V, l, w, h.)

11. The volume of a cube equals the cube of one edge. (V, e.)

12. The volume of a regular pyramid equals one third of the area of its base times its height. (V, B, h.)

13. The area of a curved surface of a right circular cylinder equals 2π times its radius times its height. (S, r, h.)

14. The volume of a right circular cylinder equals π times the square of its radius times its height. (V, r, h.)

15. The area of the curved surface of a right circular cone equals π times its radius times its *slant* height. (S, r, l.)

16. The volume of a right circular cone equals $\frac{1}{3}\pi$ times the square of its radius times its height. (V, r, h.)

17. The area of the surface of a sphere equals 4π times the square of its radius. $(S, r.)$

18. The volume of a sphere equals $\frac{4}{3}\pi$ times the cube of its radius. $(V, r.)$

19. The area of a circular ring equals the difference between the areas of the two circles inclosing the ring. $(A, r_1, r_2.)$

20. Write a formula for the volume of a hollow cylinder. $(V, r_1, r_2, h.)$

§ 11. Formulas of Science and Industry.

EXERCISES

Write a formula for each of the following laws, or rules, explaining each letter used.

1. The distance passed over by a body in uniform motion equals the rate times the time. $(d, r, t.)$

2. The space passed over by a falling body equals 16.08 times the square of the time. $(s, t.)$

NOTE. In order to write the formula for Ex. 2, and the other scientific facts that follow, it is not essential that you understand the subject matter connected with the facts.

3. The law of the lever is: The product of one force by its distance from the fulcrum equals the product of the other force by its distance from the fulcrum. $(f_1, d_1, f_2, d_2.)$

4. The density of a substance equals the ratio of its weight to its volume. $(D, W, V.)$

5. The horse-power of a gasoline engine is found by multiplying the square of the diameter (in inches) of one cylinder by the number of cylinders, and dividing that product by 2.5. $(h.p., d, n.)$

6. Ohm's Law for electric current is: The intensity of the current (in *amperes*) equals the quotient of the electromotive force (in *volts*) divided by the total resistance (in *ohms*). $(I, E, R.)$

7. The law for changing the reading on the centigrade thermometer to the reading on the Fahrenheit thermometer is: The Fahrenheit reading equals nine fifths ($\frac{9}{5}$) the centigrade reading plus 32°. $(F, C.)$ (See Fig. 5.)

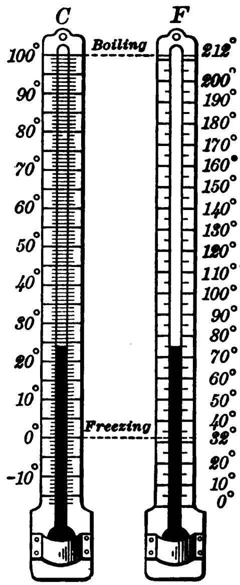

FIG. 5.

8. The law for changing from Fahrenheit reading to centigrade reading is: The centigrade reading equals five ninths of the remainder obtained by taking 32° from the Fahrenheit reading.

9. The surface speed of a wheel (in feet per minute) equals π times the diameter (in inches) times the number of revolutions per minute divided by 12. $(F, d, R.)$

10. The horse-power of a steam engine is found in the following way. Find the product of the four quantities, mean effective steam pressure (in pounds per square inch), length of piston stroke (in feet), area of the piston (in square inches), and number of strokes per minute; divide this product by 33,000. $(H. P., P, L, A, N.)$

11. The law of the jackscrew is: The ratio of the weight to be lifted to the force required equals the ratio of the circumference of the circle traced by the end of the

handle to the pitch of the screw (the distance between two successive threads of the screw. See Fig. 6.). (*W*, *F*, *c*, *p*.)

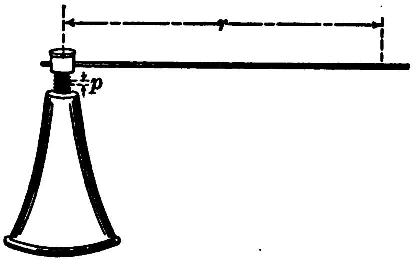

FIG. 6.

§ **12. Applications.** The problems in this section are solved by the aid of the formulas constructed in sections 10 and 11. All formulas of mensuration will be found in Part II, Chapter XXI.

When solving problems by the use of formulas, take the following steps:

(1) Write the formula.

(2) Substitute the known values in the formula.

(3) Estimate the result and record it with its proper label.

(4) Compute the result and record it with its proper label.

(5) Note how nearly your estimated and computed results agree.

EXAMPLE. A cylindrical can is 8.0″ high and 5.0″ in diameter. Find its contents.

SOLUTION.	$V=\pi r^2h$	Estimate
	$V=3.14\times(2\frac{1}{2})^2\times 8$	$V=150$ cu. in.
	$V=157$	*Ans.* 157 cu. in.

PROBLEMS

1. Find the perimeter and the area of a rectangle, if its base is 7.5″ and its height is 5.2″.

2. Find the perimeter and area of a square, if its side is $3\frac{1}{4}$″.

3. Find the area of a triangle, if its base is $8\frac{1}{2}$″ and its height is 5.4″.

4. Find the area of a trapezoid, if its bases are $7\frac{1}{4}$″ and $8\frac{1}{4}$″ and its height is 9.40″.

5. Find the circumference and area of a circle whose diameter is 9.0″. (Use 3.14 for the value of π.)

6. In a certain right triangle the two sides that include the right angle are 60″ and 11″. Find its hypotenuse.

7. The area of a rectangle is 48.3 sq. in. and its length is 11.5″. Find its width.

8. A square lot of land has an area of 576 sq. ft. How many feet of fence will be required to inclose it?

9. A square lot of land has an area of 420 sq. ft. How many feet of fence will be required to inclose it?

10. The perimeter of a square and the circumference of a circle are of the same length, 42″ each. Which incloses the larger area? How much larger? What per cent larger?

11. In a certain city there are 80 miles of street railways. The wire carrying the electric current has a $\frac{3}{8}$″ diameter. How many cubic feet of wire are being used to carry the current?

(Suggestion. The wire is a cylinder. See that the units are of the same kind before using the formula.)

12. A certain steel ball is 3″ in diameter.

(a) What is its volume?

(*b*) What is its weight, if a cubic inch of the steel weighs .28 lb.?

13. Find the equal sides of an isosceles triangle, if its base is 20″ and its height 12″. (Fig. 7.)

NOTE. *The altitude of an isosceles triangle bisects the base.*

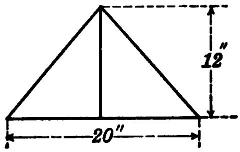

FIG. 7.

14. Find the altitude and area of an equilateral triangle, if each side is 12″. (Fig. 8.)

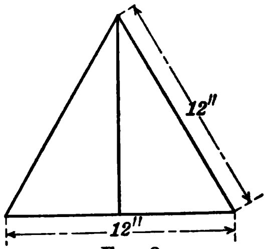

FIG. 8.

15. A baseball diamond is a square 90′ on a side. What is the distance from first base to third base? From second base to the "home plate"?

16. A circle has a 4″ diameter. How long is an arc of 40°? (SUGGESTION. An arc of $40° = \frac{40}{360}$ or $\frac{1}{9}$ of the circumference.)

17. Find the length of a 60° arc on a 12″ circle.

NOTE. A 12″ circle means a circle having a 12″ diameter.

18. A belt is in contact with a 36″ pulley for 200° of the circumference. What is the length of the arc of contact? (Fig. 9.)

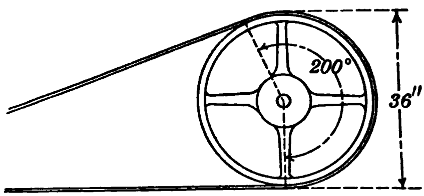

FIG. 9.

19. The arc of contact of a belt on a 15″ pulley is 170°. Find the length of the belt in contact with the pulley.

20. A belt connects two 18″ pulleys and the centers of the pulleys are 10′ 6″ apart. How long is the belt? (Fig. 10.)

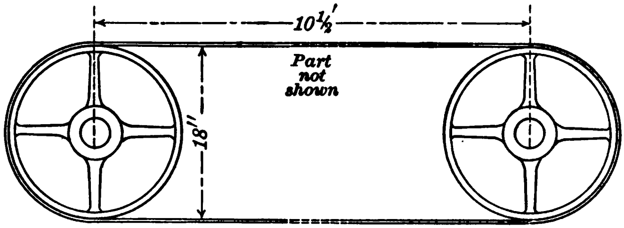

FIG. 10.

21. What is the horse-power of the engine of an automobile having 6 cylinders, each 4″ in diameter? (See Ex. 5, page 37.)

22. Find the horse-power of a 4-cylinder engine, each cylinder being $4\frac{1}{4}''$ in diameter.

23. Make the following changes in thermometer readings:

(*a*) Fahrenheit readings for 45° C.; 100° C.; 27° C.; 10° C.; 0° C. (See Ex. 7, page 38.)

(*b*) Centigrade readings for 77° F.; 212° F.; 67° F.; 40° F.; 32° F. (See Ex. 8, page 38.)

24. Find the surface speed of a grindstone if its diameter is 30″ and if it is running at 90 revolutions per minute. (See Ex. 9, page 38.)

CHAPTER IV

METRIC MEASURES

§ 13. **Metric Measures.** The metric measures are used by nearly all the countries of continental Europe and South America. They were first used in France shortly after the French Revolution. With the increasing business relations with these countries it is necessary that we become familiar with the measures used by them. The metric measures are also used in scientific laboratories. When you become familiar with them, you will see that they are much more convenient to use than our measures.

§ 14. **Linear Measure.** The unit of linear measure is the ***meter***. The meter = 39.37″, a little more than a yard.

The meter (*m.*), is divided into 100 equal parts, called ***centimeters*** (*cm.*). Each centimeter is again divided into 10 equal parts, called ***millimeters*** (*mm.*). For long distances the ***kilometer*** (*Km.*) is used. The kilometer = 1000 meters and is about .6 of a mile. The four measures named above are the ones commonly used.

TABLE

10 millimeters (mm.)	= 1 centimeter (cm.)	= .3937 in.
100 centimeters	= 1 meter (m.)	= 39.37 in.
1000 meters	= 1 kilometer (km.)	= .621 mi.

The tables for metric measures are on page 292.

EXERCISES

1. Express a meter in feet to the nearest tenth of a foot. Express a kilometer in feet.

2. What is the diameter, in inches, of a 42-centimeter gun? A 310-millimeter gun? A 75-millimeter gun? A 36-centimeter gun?

3. An army captures 700 meters of trenches. Express this distance in yards.

4. An army advances $4\frac{1}{2}$ kilometers. How many miles does it advance?

5. The wing-spread of a certain 80-horse-power French biplane is 28 meters. Express this in feet.

6. The wing-spread of a certain 220-horse-power French biplane is 13 meters. Express this in feet.

7. To become a pilot of a scouting airplane "at the front," one of the requirements was that the aviator must make an altitude of 7000 meters. Express this height in kilometers; in miles.

8. The unit of area is the square meter. How many square centimeters does it contain?

9. The unit of volume is the cubic meter. How many cubic centimeters does it contain?

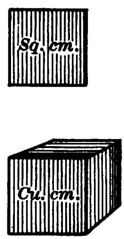

Fig. 11.

10. Measure your desk cover in centimeters and find its area.

11. Measure the length and width of the floor of the classroom in meters. Find its area.

12. How much longer is a 100-meter dash than a 100-yard dash?

13. Suppose you walk at the average rate of $3\frac{1}{2}$ miles per hour. What is your average rate in kilometers per hour?

14. What is your height in centimeters? In meters?

15. A group of French boys wish to lay out a baseball diamond of the same size as the American diamond.

(*a*) How long, in meters, will they make each side of the square? (See Ex. 15, page 41.)

(*b*) The pitcher's box is 60.5′ from the home plate. How far, in meters, will they locate the pitcher's box from the home plate?

(*c*) In making a home run, what part of a kilometer will be traveled?

(*d*) What is the area of the baseball diamond in square meters?

16. Figure 12 is the plan of a double tennis court, drawn to the scale of 40 feet to the inch.

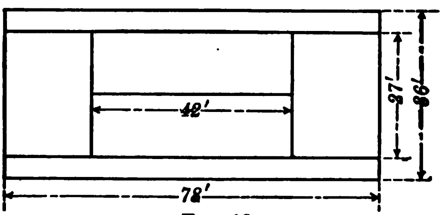

FIG. 12.

(*a*) How many meters would a French boy make each of these distances, if he were laying out a court?

(*b*) Draw a plan of this tennis court to the scale of 20 centimeters to one centimeter.

(*c*) How many square meters are there in the tennis court?

§ 15. **Weight.** The metric unit of weight is the ***gram***. The gram is the weight of *one cubic centimeter of water*. It is a little more than .03 of an ounce, so you see that it is a very small weight. The weight that is in most common use, corresponding to our pound, is the ***kilogram*** (1000 grams); it is equivalent to very nearly 2.2 pounds.

TABLE

	1 gram (g.)	= .03527 oz.
1000 grams	= 1 kilogram (Kg.)	= 2.20462 lb.
1000 kilograms	= 1 metric ton (T.)	= 2204.62 lb.
		= 1.102 U. S. tons.

EXERCISES

1. An order for 5 kilograms of sugar would be for how many pounds?

2. A grocer receives the following order:

$3\frac{1}{2}$ Kg. beef
12 Kg. potatoes
$\frac{1}{2}$ Kg. pork
200 g. pepper

How would he fill it using pounds and ounces?

3. What is the weight in pounds of a 700-kilogram shell?

4. How many kilograms of flour are there in a barrel? (1 bbl. flour = 196 lb.)

5. How much does a ton of coal weigh in kilograms?

6. A certain British war tank weighs 75 metric tons. What is its weight in U. S. tons?

7. A certain type of steel merchant vessel weighs 8000 metric tons. What is its weight in U. S. tons?

8. What is your weight in kilograms?

9. Express in kilograms the weight of the United States' standard bushel of each of the following grains: (*a*) wheat, 60 lb., (*b*) corn, shelled, 56 lb., (*c*) barley, 48 lb., (*d*) oats, 32 lb.

10. What price per kilogram is equivalent to each of the following prices per bushel: (*a*) oats, $1.00, (*b*) barley, $1.75, (*c*) corn, shelled, $2.00, (*d*) wheat, $2.20?

§ 16. **Capacity.** The metric unit of capacity is the ***liter***. The liter is equivalent to very nearly 1.06 quarts (liquid measure).

1000 cubic centimeters = 1 liter (l.)

Since one cubic centimeter of water weighs 1 gram, a liter of water weighs 1 kilogram.

EXERCISES

1. What is the weight of a liter of water in pounds?

2. A gallon jug holds how many liters?

3. If milk is selling at 12 cents a quart, what will 5 liters of milk cost?

4. A liter of milk is sold for 15 cents. How much will eight quarts cost?

5. The capacity of the gasoline tank on a certain automobile is 10 gallons. What is the capacity of the tank in liters?

6. What price per liter is equivalent to 28 cents per gallon for gasoline?

§ 17. Specific Gravity. One cubic centimeter of water weighs one gram. One cubic centimeter of mercury weighs 13.6 grams; that is, mercury is 13.6 times as heavy as water. This result, 13.6, is known as the ***specific gravity*** of mercury.

The ***specific gravity*** of a substance is the ratio of the weight of a certain volume of the substance to the weight of the same volume of water. For example, the weight of one cubic foot of a certain kind of brick is 125 pounds and the weight of one cubic foot of water is 62.5 pounds, hence the specific gravity of brick is $125 \div 62.5$, or 2.

EXERCISES

1. Explain the statement: The specific gravity of a certain kind of steel is 7.8.

2. A block of steel (Ex. 1.) contains 200 cu. cm. What is its weight in grams? In kilograms?

3. The specific gravity of a certain kind of maple wood is 0.8. What is the weight in kilograms of a beam 5 m. long, 10 cm. wide, and 8 cm. thick?

4. A certain rectangular iron plate is 1 cm. thick, 18 cm. long, and 3 cm. wide. How many grams does it weigh? (The specific gravity of the iron is 7.2.)

5. A circular iron plate is $\frac{1}{2}$ cm. thick and 30 cm. in diameter. Find its weight in grams, if the specific gravity is 7.2.

6. A block of iron is 28 cm. long, 15 cm. wide, and 10 cm. thick. How many kilograms does it weigh if its specific gravity is 7.1? How many pounds?

7. The specific gravity of a certain kind of copper is 8.9. Find the weight of a solid copper cylinder 15 cm. high and 10 cm. in diameter.

8. Find the weight of a copper block 12 mm. high, 14 mm. wide, and 20 mm. long, if its specific gravity is 8.9.

9. The specific gravity of mercury is 13.6. Find the weight of the mercury that fills a circular tube 5 mm. inside diameter and 20 cm. long.

10. The specific gravity of ice is .92. What is the weight of a block of ice 24 cm. by 18 cm. by 10.5 cm.?

11. In the metric system the density of a certain kind of iron is 7.3 g. per cubic centimeter. The specific gravity of the iron is 7.3. Is the density of the iron 7.3 lb. per cubic foot?

12. The density of water is 62.5 lb. per cubic foot. What is the density of the iron in Ex. 11, in lb. per cubic foot?

CHAPTER V

LINEAR EQUATIONS

§ 18. **Equations.** An equation is a statement of the equality of two number expressions. The two number expressions are called the members of the equation.

There are two kinds of equations, the *equation of condition* and the *identity*.

The ***equation of condition*** is an equation in which the members are equal only when the letters have particular values. For example, $5x=10$ is an equation, the particular value of x being 2. Equations of condition are the equations used in the solution of problems.

The ***identity*** is an equation in which the members merely represent different ways of writing the same number. For example, $5x+3x\equiv 8x$ is an identity.

In expressing identities involving *letters*, the symbol $\equiv$ is often used instead of the symbol $=$. The symbol $\equiv$ is read "is identical to."

In Chapter VII, Second Course, equations were solved by using one or more of the following axioms:

(1) *If the same number is added to equal numbers, the sums are equal.*

(2) *If the same number is subtracted from equal numbers, the remainders are equal.*

(3) *If equal numbers are multiplied by the same number, the products are equal.*

(4) *If equal numbers are divided by equal numbers, the quotients are equal.*

The *root* of an equation is the value of the unknown quantity which *satisfies* the equation. The *check* shows whether or not the value obtained from the solution is the root.

In checking the root of an equation, the following axiom is used:

(5) *A number may be substituted for its equal in an equation.*

EXAMPLE 1. Solve the equation $\frac{3a}{2}=\frac{9}{4}$, and check the answer.

SOLUTION.

① $\frac{3a}{2}=\frac{9}{4}$

② $6a=9$ ①×4

③ $a=1.5$ ②÷6

CHECK.

$$\frac{3\times1.5}{2}\overset{?}{=}\frac{9}{4}$$

$$\frac{4.5}{2}\overset{?}{=}\frac{9}{4}$$

$$2.25=2.25$$

Ans. $a=1.5$.

In checking the root of an equation, the following rules should be observed:

(1) Substitute the value of the root obtained in the original equation, — not in any subsequent step.

(2) Simplify each member of the equation by itself.

(3) Retain the question mark over the equality sign in the check until the two members of the equation are shown to be equal. In case the value of the root is an approximation, the two members can be shown to be only approximately equal; hence, retain the sign $\overset{?}{=}$ throughout.

EXAMPLE 2. Solve the equation $5n-6=2n+12$, and check the answer.

SOLUTION.

①	$5n-6=2n+12$	
②	$5n=2n+18$	①+6
③	$3n=18$	②−$2n$
④	$n=6$	③÷3

The symbol, ①+6, means that 6 is added to each member of equation ①.

The symbol, ②−$2n$, means that $2n$ is subtracted from each member of equation ②.

CHECK.

$$(5\times 6)-6 \stackrel{?}{=} (2\times 6)+12$$
$$30-6 \stackrel{?}{=} 12+12$$
$$24=24$$

Ans. $n=6$.

Is 6 the root of the equation $5n-6=2n+12$? How do you know?

EXERCISES

Solve each of the following equations and check each answer.

1. $3m=12$
2. $5a+3a=16$
3. $3y+y=10$
4. $7y-5y=8$
5. $4m+3m=21$
6. $3a+2a+a=24$
7. $7b+5b-8b=16$
8. $5x+2x-x=36$
9. $a+8=11$
10. $x+3=7$
11. $14=m+11$
12. $28=a+15$
13. $2y+3=7$
14. $2b+12=17$
15. $3b+4=16$
16. $15=10b+10$
17. $38=2c+9$
18. $y-3=8$

19. $4=m-5$

20. $3a-4=8$

21. $5b-3=7$

22. $4k-6=16$

23. $44=8c-4$

24. $5a-2=a+26$

25. $3n+2=n+8$

26. $5d+7=9d+3$

27. $4k+13=6k-11$

28. $5w+3=9w-17$

29. $5a+6=7a-8$

30. $7w-2=4w+19$

31. $3x+2=27-2x$

32. $7y-4=16-3y$

33. $15-2y=3y+10$

34. $5w-6=40-3w$

35. $\frac{y}{6}=\frac{23}{3}$

36. $\frac{2m}{3}=12$

37. $\frac{5b}{2}=12$

38. $\frac{4m}{9}=10$

39. $\frac{2w}{3}=\frac{8}{5}$

40. $\frac{5k}{4}=.8$

41. $\frac{5}{6}=\frac{2w}{15}$

42. $\frac{2y}{3}=\frac{3}{5}$

43. $\frac{10}{x}=2$ (Multiplying by x, $10=2x$.)

44. $\frac{12}{y}=4$

45. $\frac{6}{m}=4$

46. $\frac{4}{c}=3$

47. $\frac{3}{d}=12$

48. $4.8=\frac{12}{r}$

49. $\frac{9}{2x}=1.5$ (Multiplying by $2x$, $9=3x$.)

50. $\frac{42}{4y}=3$

51. $5=\frac{50}{2.5y}$

52. $2=\frac{18}{5y}$

53. $\frac{4}{w}=\frac{1}{2}$ (Multiply by $2w$.)

54. $\frac{5}{2}=\frac{10}{y}$

55. $\frac{3}{x}=\frac{5}{12}$

56. $\frac{15}{p}=\frac{45}{7}$

57. $\frac{2}{5m}=\frac{4}{15}$ (Multiply by $15m$.)

58. $\frac{3}{10k}=\frac{15}{100}$

59. $\frac{2}{5}=\frac{4}{5t}$

60. $\frac{2.4}{12}=\frac{1.8}{4x}$

61. $3(2b+3)=21$

(Removing parentheses, $6b+9=21$.)

62. $4(m-2)=20$

63. $3(d+7)=45$

64. $15=3(x-1)$

65. $12=3(y+1)$

66. $2(3k+4)=26$

67. $5(b+2)=2(b+14)$

68. $4(m-2)=3(m+1)$

69. $8(a+5)=2(3a+26)$

70. $5(2y+1)=3(4-y)$

PROBLEMS

1. The average rate per hour that a certain boy travels on his bicycle is 15 miles. This is one more mile than four times his average rate when walking. What is his average rate when walking?

SOLUTION. Let $r=$ the boy's average rate in miles per hour when walking, then $4r+1=$ his average rate on his bicycle.

The equation is:

①	$4r+1=15$	
②	$4r=14$	①-1
③	$r=3\frac{1}{2}$	②$\div 4$

CHECK. Substitute, in the statement of the given problem, $3\frac{1}{2}$ miles per hour for the average rate of the boy when walking. *Ans.* $3\frac{1}{2}$ miles per hour.

2. By vote of the United States Senate on the Food Control Bill, August 8, 1917, the bill was carried by a majority of 59. The total number of senators voting was 73. (*a*) How many voted for the bill? (*b*) What per cent voted for the bill?

3. On a certain Y. M. C. A. track a boy has to go 10 laps and 100 yd. besides, for a mile run. (*a*) What is the distance around the track? (*b*) What per cent of a mile is one lap?

4. The bill for the Suffrage Federal Amendment was carried by a majority of 138 in the House of Representatives on January 10, 1918. The total number of representatives voting was 410. (*a*) How many voted for the bill? (*b*) What per cent?

5. A freight train running 22 miles an hour is 154 miles ahead of an express train running 50 miles an hour. In how many hours will the express overtake the freight?

6. A transport averaging 21 knots an hour leaves a certain port when a merchant ship, averaging 17 knots an hour, is already 160 knots out. In how many hours will the transport overtake the merchant ship?

7. A fleet averaging 15 knots an hour is 630 knots from a certain port when a destroyer, averaging 22 knots an hour, starts out to overtake it. In how many hours will the destroyer overtake the fleet?

8. A, traveling in an airplane at 95 miles an hour, sets out to overtake B who is traveling in an airplane at 80 miles an hour. If B had a start of 135 miles, in how many hours will A overtake him?

9. A certain 12″ I-beam used in building construction weighs 525 lb. The density of the iron is 450 lb. per cubic foot. How many cubic feet of iron are there in the beam?

(SUGGESTION. Use the formula $D=\frac{W}{V}$.)

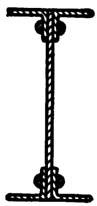

FIG. 13.

10. A certain piece of steel weighs 16.8 lb. The density of the steel is .28 lb. per cubic inch. How many cubic inches are there in the piece?

11. A certain ball of cork weighs 35 lb. The density of cork is 16 lb. per cubic foot. How many cubic feet are there in the ball?

12. A part of the trunk of a certain chestnut tree weighs 148 lb. The density of the chestnut is 33 lb. per cubic foot. Find its volume in cubic feet.

13. The sum of two consecutive integers is 47. Find them.

SOLUTION. Let $n =$ the smaller integer, then
$n+1 =$ the larger integer, and
$2n+1 =$ their sum.

The equation is:

①	$2n+1=47$	
②	$2n=46$	① -1
③	$n=23$	② $\div 2$
④	$n+1=24$	③ $+1$

CHECK. Substitute 23 and 24 for the two integers. The work is left to the student. *Ans.* 23 and 24.

14. In a certain number of two digits, the units' digit is 3 more than the tens' digit. When the number is divided by 4, the quotient is 9. Find the number.

[Thus, 47 is an illustration of a number of two digits; 4 is the tens' digit and 7 is the units' digit. To form the number, you multiply the tens' digit by 10 and add the units' digit; for example, $4 \times 10 + 7 = 47$.]

SOLUTION. Let t = the tens' digit, and
$t+3$ = the units' digit, then
$10t+(t+3)$ = the number.

The equation is:

①	$\frac{10t+t+3}{4}=9$	
②	$10t+t+3=36$	①×4
③	$11t+3=36$	②≡
④	$11t=33$	③−3
⑤	$t=3$	④÷11
⑥	$t+3=6$	⑤+3

The number is: $3\times 10+6$, or 36.

CHECK. Substitute 36 for the number. The work is left to the student. *Ans.* 36.

15. There are two consecutive integers whose sum is 75. Find them.

16. There are three consecutive integers whose sum is 246. Find them.

17. There are two consecutive integers whose sum increased by 5 is equal to 42. Find them.

18. There are two consecutive integers whose sum decreased by 4 is equal to 45. Find them.

19. There are two consecutive integers whose sum divided by 7 gives a quotient of 9. Find them.

20. If you add 3 to a certain number and then divide this sum by 7, the quotient will be 13. Find the number.

21. If you subtract 3 from a certain number and then divide this result by 3, the quotient will be 27. Find the number.

22. If a certain number is divided by 5, the quotient is 8 and the remainder is 3. Find the number.

(The equation is: $\frac{n}{5}=8+\frac{3}{5}$, or $\frac{n-3}{5}=8$. Why?)

23. If a certain number is divided by 7, the quotient is 20 and the remainder is 4. Find the number.

24. There are two consecutive integers whose sum divided by 6 gives a quotient of 9 and a remainder of 3. Find the integers.

25. If 9 is added to a certain number and this sum is divided by 7, the quotient is 8. Find the number.

26. There are two *consecutive even* numbers whose sum is 42. Find them.

27. There are two *consecutive odd* numbers whose sum diminished by 2 is 30. Find them.

28. If 4 is added to twice a certain number and the result is divided by 5, the quotient is 3 and the remainder is 1. Find the number.

29. In a certain number of two digits, the units' digit exceeds the tens' digit by 2. The number is 4 times the sum of its digits. Find the number.

30. In a certain number of two digits, the tens' digit is twice the units' digit. The number is 12 more than 5 times the sum of its digits. Find the number.

31. In a certain number of two digits, the units' digit exceeds the tens' digit by 5. When the number is divided by 3, the quotient is equal to the sum of the digits. Find the number.

32. In a certain number of two digits, the tens' digit exceeds the units' digit by 5. If 6 is subtracted from the number and the result is divided by 6, the quotient is 2 more than the sum of the digits. Find the number.

§ 19. Transformation of Formulas. The formula for the area of the circle is: $A=\frac{\pi d^2}{4}$. A is called the *subject* of the formula, just as in the *rule* for the area of the circle, the word "area" is called the subject of the sentence. *To transform a formula* means to change the *subject* of the formula. This requires the solution of the given formula for the new *subject*.

When it is necessary to find the diameters of circles which shall have given areas, time and labor can be saved by changing the subject of the formula for the area of the circle from A to d.

Example. Change the subject of the formula $A=\frac{\pi d^2}{4}$ from A to d.

Solution.

①	$A=\frac{\pi d^2}{4}$	
②	$4A=\pi d^2$	①×4
③	$\frac{4A}{\pi}=d^2$	②÷π
④	$\sqrt{\frac{4A}{\pi}}=d$	③$\sqrt{\ }$
⑤	$d=\sqrt{\frac{4A}{\pi}}$	④≡

Ans. $d=\sqrt{\frac{4A}{\pi}}$.

EXERCISES

Change the subject of each of the following formulas as required.

1. $A = bh,\ b = ?$
2. $A = \frac{1}{2}bh,\ h = ?$
3. $A = s^2,\ s = ?$
4. $c = \pi d,\ d = ?$
5. $c = 2\pi r,\ r = ?$
6. $d = rt,\ r = ?$
7. $w_1d_1 = w_2d_2,\ w_1 = ?$
8. $w_1d_1 = w_2d_2,\ d_2 = ?$
9. $g = \frac{r}{h},\ r = ?$
10. $i = prt,\ p = ?$
11. $A = \pi r^2,\ r = ?$
12. $V = lwh,\ w = ?$
13. $S = 2\pi rh,\ r = ?$
14. $V = \pi r^2h,\ h = ?$
15. $V = \pi r^2h,\ r = ?$
16. $S = \pi rl,\ l = ?$
17. $V = \frac{1}{3}Bh,\ B = ?$
18. $V = \frac{1}{3}\pi r^2h,\ h = ?$
19. $V = \frac{1}{3}\pi r^2h,\ r = ?$
20. $S = 4\pi r^2,\ r = ?$
21. $V = \frac{4}{3}\pi r^3,\ r = ?$
22. $A = \frac{1}{2}(b_1 + b_2)h,\ h = ?$
23. $\angle A + \angle B = 180,\ \angle A = ?$
24. $c^2 = a^2 + b^2,\ a = ?$
25. $c^2 = a^2 + b^2,\ b = ?$
26. $H = \frac{d^2n}{2.5},\ d = ?$
27. $I = \frac{E}{R},\ E = ?$
28. $\frac{W}{F} = \frac{c}{p},\ F = ?$
29. $S = \frac{1}{2}\,gt^2,\ t = ?$
30. $F = \frac{9}{5}C + 32,\ C = ?$

§ 20. Graphs of Linear Equations. The equation $x + y = 4$ contains two unknown numbers. One solution of this equation is $x = 1,\ y = 3$; a second solution is $x = 2,\ y = 2$; a third solution is $x = 3,\ y = 1$; etc. The value obtained for x depends upon the value given to y; that is, x changes with y or *varies* with y.

In Chapter VIII, Second Course, you learned how to locate these pairs of values on squared paper, and you found that, when you joined the points thus located, you had a *straight line*, as in Fig. 14. This is the ***graph*** of the equation $x+y=4$.

EXERCISES

1. From the equation $x+y=4$, when $x=3$, $y=$? Which point does this pair of values of x and y locate on the graph?

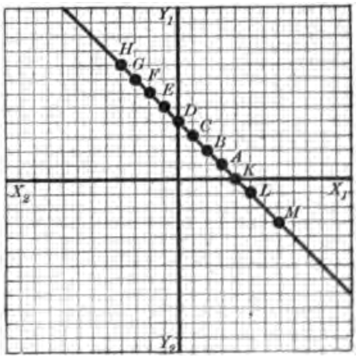

Fig. 14.

2. From the equation when $x=4$, $y=$? Which point does this pair of values locate?

3. What value of y is paired with $x=+5$? Which point is it?
Ans. $y=-1$, Point L.

4. What value of y is paired with $x=+7$? Which point is it?

5. What value of x is paired with $y=+5$? Which point is it?

6. What value of x is paired with $y=+6$? Which point is it?

7. What value of x is paired with $y=+7$? Which point is it?

8. What value of x is paired with $y=+8$? Which point is it?

NOTE. From exercises 1–8 it appears that:

(*a*) *All* values of x to the right of the axis Y_1Y_2 are ***plus***, so *all* values of x to the left of Y_1Y_2 are ***minus***.

(*b*) *All* values of y above the axis X_1X_2 are ***plus***, so *all* values of y below X_1X_2 are ***minus***.

9. Show that all pairs of values obtained in Exs. 3–8 satisfy the equation $x+y=4$.

10. What are the pairs of values for the following points in Fig. 14?

(*a*) Point M?
(*b*) Point D?
(*c*) Point E?
(*d*) Point A?
(*e*) Point G?

11. Find four pairs of values for the equation $x+2y=9$. Let $y=1$, 2, 3, and 4, and tabulate these values.

x	?	?	?	?
y	1	2	3	4

12. Locate on squared paper the four points for the pairs of values obtained in Ex. 11. Draw the graph.

13. From the graph fill in the following table.

x	−1	−3	−5	?	?	?
y	?	?	?	+7	+8	0

14. Find out whether or not the pairs of values in the table in Ex. 13 satisfy the equation $x+2y=9$.

CHAPTER VI

POSITIVE AND NEGATIVE NUMBERS

§ 21. **Signed Numbers.** In checking the pairs of values in the equations in the last section you were dealing with numbers having + and − signs in front of them. Such numbers are ***signed*** numbers. Those having plus signs in front of them are ***positive*** numbers; those having minus signs in front of them are ***negative*** numbers.

On the squared paper all values of x to the right of the Y_1Y_2 axis (Fig. 14) and all values of y above the X_1X_2 axis are *positive;* all values of x to the left of the Y_1Y_2 axis and all values of y below the X_1X_2 axis are *negative.* Hence, when numbers are represented by points on lines, negative numbers are measured in the *opposite* direction from positive numbers. In general, the negative number is opposite in quality to the positive number.

CONCRETE ILLUSTRATIONS OF IDEAS OPPOSITE IN QUALITY

POSITIVE	NEGATIVE
Cash on hand	Bills to be paid
Gain	Loss
Assets	Liabilities
Rise in temperature	Fall in temperature
Direction East	Direction West
Direction North	Direction South

§ 22. Addition of Signed Numbers of the Same Quality.

EXAMPLE 1. (*a*) A man gains \$300 and later gains \$200 more. What is the result?

A gain is expressed by a positive number, hence combining $+300$ and $+200$, we get $+500$.

Ans. \$500 *gain.*

EXAMPLE 2. A man loses \$300 and later loses \$200 more. What is the result?

A loss is expressed by a negative number, hence combining -300 and -200, we get -500. *Ans.* \$500 *loss.*

EXAMPLE 3. (*a*) The temperature rises 12° and later rises 15° more. What is the total change?

A rise in temperature is expressed by a positive number, hence combining $+12$ and $+15$, we get $+27$.

Ans. 27° *rise.*

EXAMPLE 4. The temperature falls 12° and later falls 15° more. What is the total change?

A fall in temperature is expressed by a negative number, hence combining -12 and -15, we get -27.

Ans. 27° *fall.*

EXERCISES

Combine the following pairs of numbers, placing the necessary sign in front of the answer.

1. $+7$ and $+4$
2. -3 and -4
3. -9 and -15
4. $+13$ and $+3$
5. $+4$ and $+1$
6. -7 and -7
7. -15 and -1
8. $+3$ and $+7$
9. -2 and -1
10. -40 and -3

11. $+7$ and $+9$

12. -7 and -8

13. $+15$ and $+8$

14. -12 and -18

15. -1 and -1

16. $+2\frac{1}{2}$ and $+5\frac{1}{2}$

17. $-\frac{1}{2}$ and $-\frac{1}{4}$

18. $+\frac{1}{2}$ and $+\frac{1}{3}$

19. $-3\frac{1}{2}$ and $-5\frac{3}{8}$

20. $+12\frac{1}{4}$ and $+6\frac{7}{8}$

NOTE. In each of the exercises 1–20 the signs of the numbers to be combined are the *same*. In each exercise the *sum* of the two numbers is obtained and the sign of the answer is the same as the sign of the two numbers.

21. A ship travels north 80 miles the first day, 72 miles the second and 68 miles the third. Find the average distance per day and express the result as a signed number.

22. A man owes bills of $25, $32.50, $17.20, and $8.90. Find his total debts and express the result as a signed number.

23. A provision dealer has five unfilled orders valued at $4.50, $8.75, $18.25, $6.80, and $11.40. Find the total amount of unfilled orders and express the result as a signed number.

24. The temperature readings taken every three hours for a certain day were: $+36°$, $+42°$, $+50°$, $+68°$, $+72°$, $+65°$, $+57°$, $+48°$. Find the average temperature for the day.

25. The temperature readings taken every three hours for a certain day were: $-12°$, $-10°$, $-6°$, $-4°$, $-1°$, $-5°$, $-6°$, $-4°$. Find the average temperature for the day.

§ 23. Addition of Signed Numbers of Opposite Quality.

Example 1. During a certain day the temperature rises 10° and then falls 25°. What is the direct change in the temperature for the whole day?

A fall in temperature is *opposite in quality* to a rise. A rise of 10°, written $+10$, and a fall of 25°, written -25, when combined, give a fall of 15°, written -15.

Ans. 15° *fall.*

Example 2. A man gains \$500 and then loses \$700. What is the net result?

A loss of money is *opposite in quality* to a gain. A gain of \$500, written $+500$, and a loss of \$700, written -700, when combined, give a loss of \$200, written -200.

Ans. \$200 *loss.*

Example 3. A man loses \$350 and gains \$750. What is the net result?

Combining -350 and $+750$, we get $+400$, the plus number being the larger. *Ans.* \$400 *gain.*

Example 4. A motorcyclist travels 40 miles east and then 65 miles west. How far is he from his starting point and in which direction?

Direction east is expressed by a positive number, hence combining $+40$ and -65, we get -25, the minus number being the larger. *Ans.* 25 miles west.

In combining a positive number with a negative number:

(1) Note which is the larger, the positive or the negative number.

(2) Note how much larger it is.

(3) Note which sign is to be placed in front of the answer.

EXERCISES

Combine the following pairs of numbers, placing the necessary sign in front of the answer.

1. $+12$ and -7
2. $+10$ and -16
3. -4 and $+8$
4. -3 and $+7$
5. $+8$ and -3
6. $+15$ and -11
7. $+9$ and -12
8. $+8$ and -17
9. -10 and $+3$
10. -15 and $+1$
11. -4 and $+4$
12. -12 and $+15$
13. -9 and $+12$
14. -12 and $+14$
15. -10 and $+5$
16. $+14$ and -2
17. $+11$ and -18
18. $+32$ and -33
19. $+12$ and -12
20. -1 and $+9$

NOTE. In each of the exercises 1–20 the signs of the numbers to be combined are *opposite*. In each exercise the *difference* between the two numbers is obtained and the sign of the answer is the sign of the larger number.

21. The temperature at 1.00 P.M. on a certain day was $+12°$. During the next six hours it fell 15°. What was the temperature at 7.00 P.M.?

22. A man loses \$35 on one transaction and gains \$47 on another. What is the net result? Express the result as a signed number.

23. An automobilist travels 45 miles east and then 120 miles west. How far is he from his starting point? Express the result as a signed number.

24. Emperor Augustus Cæsar was born in 63 B.C. and died when he was 77 years old. What was the date of his death? Express the result as a signed number.

25. The water in a reservoir rises 5 in., then falls 7 in., rises again 12 in., then falls again 9 in. Find the net rise or fall. Express the result as a signed number.

26. The midnight temperatures for a certain week were as follows: $+8°$, $-4°$, $-9°$, $+1°$, $+12°$, $+8°$, $+6°$. Find the average of these temperatures.

27. The midnight temperatures for the next week were as follows: $+4°$, $0°$, $-3°$, $-8°$, $-12°$, $-1°$, $+2°$. Find the average of these temperatures.

28. The latitude of New Orleans, La., is $+30°$, and of Boston, Mass., $+45°$. Find the latitude of Richmond, Va., halfway between them.

29. The latitude of Chicago, Ill., is $41° \; 50'$ and of Rio Janeiro, Brazil, $-22° \; 50'$. Find the latitude of Panama, halfway between them.

30. Pythagoras was born about 600 B.C. and died when he was 65 years old. Using signed numbers, find the probable year of his death.

31. Thales was born about 640 B.C. and died when he was 92 years old. Using signed numbers, find the probable year of his death.

32. Plato was born about 429 B.C. and died when he was 81 years old. Find the probable year of his death.

§ 24. Drill Table — Addition of Signed Numbers

	I	II	III	IV	V	VI	VII	VIII
A	+3	+3	−4	−5	3	−12	−3	3
B	−7	−8	+7	−3	−5	4	−3	−4
C	+4	+5	+3	−6	−3	3	−3	12
D	−3	−3	−6	−9	6	−6	−3	−8

Note. Any number having no sign in front of it is *positive*.

Suggested drills with this table:

(*a*) Add each number in line *B* to each number in line *A*; each number in line *C* to each number in line *A*, in line *B*; etc.

(*b*) Add each number in column I to each number in column II; in column III; etc.

(*c*) Find the sum of all the numbers in column I; in column II; etc.

(*d*) Find the sum of all the numbers in line *A*; in line *B*; etc.

Note. In adding more than two signed numbers, add the first to the second, then that sum to the third, etc.

CHAPTER VII

ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS

§ 25. **Algebraic Expressions.** ***Algebra*** as well as arithmetic deals with numbers. In algebra letters are used to represent numbers.

The formula $A=bh$, the equation $5a=10$, and the sum of two numbers $a+b$ are ***algebraic expressions***. In algebraic expressions, letters are used to represent numbers. In the product bh, b and h may be made to represent any numbers that we wish them to represent. b and h are the *factors* of the product bh.

If an expression is the product of two or more numbers, then these numbers are ***factors*** of the expression.

If an expression is the product of *two* factors, either of these factors is the ***coefficient*** of the other.

In the product bh, b is the coefficient of h, and h is the coefficient of b. In the product $5ax$, the coefficient of x is $5a$; the coefficient of a is $5x$.

EXERCISES

1. In the following products what is the coefficient of x: $5x$? $7ax$? $3abx$? $4xy$? xz? x? $5axy$?

2. The expression $2a+3y$ is the sum of $2a$ and $3y$. What is the coefficient of a? The coefficient of y?

§ 26. **Polynomials.** A ***polynomial*** is an algebraic expression composed of parts connected by the signs $+$ or $-$. Each of these parts of a polynomial, together with the sign in front of it, is a term. A polynomial of two terms

is a binomial; of three terms, a trinomial. One term by itself is a monomial.

For example, $3a+4b-5c$ is a ***trinomial***, in which the ***terms*** are $3a$, $+4b$, and $-5c$.

EXERCISES

From the following group of polynomials, select and write in separate columns the monomials, the binomials, and the trinomials: bh, $2a+3y$, $3a-2b+c$, $10a-8b$, $5c$, $2a+4$, $3x+2y-4$, $6a-2$, $4axy$, 3, $7a-3b-7$, x, $a-b-2c$, $a-2b$, $3ab-2ac$, $m+2$, $2a-3b+4$, $3x$, $a-5$.

§ 27. Exponents and Powers. The formula for the area of a square is: $A=s^2$. s^2 means $s\times s$. s^2 is read "s square" or "the second *power* of s"; the 2 is called an exponent.

The formula for the volume of a cube is: $V=e^3$. e^3 means $e\times e\times e$. e^3 is read "e cube" or "the third *power* of e"; the 3 is called an exponent.

The ***power*** of a number is the product obtained by using the number as a factor one or more times. The number is often called the ***root***, or ***base***, of the power.

The ***exponent*** of a power is the number written at the right of, and slightly above, the factor to show how many times the factor is used.

EXAMPLE 1. Write the power a^4 without an exponent. Name the exponent and the root.

SOLUTION. $a^4\equiv a\times a\times a\times a$.

In the power a^4, 4 is the exponent, a is the root.

EXAMPLE 2. Write the power 5^3 without an exponent. Name the exponent and the root.

SOLUTION. $5^3\equiv 5\times 5\times 5$.

In the power 5^3, 3 is the exponent, 5 is the root.

EXAMPLE 3. Write the product a^2x^3y without using exponents. Name the exponents and the roots.

SOLUTION. $a^2x^3y \equiv a \cdot a \cdot x \cdot x \cdot x \cdot y$.

In the power a^2, 2 is the exponent, a is the root.

In the power x^3, 3 is the exponent, x is the root.

In the power y, 1 (not written) is the exponent, y is the root.

NOTE. In algebra any letter, or any number, not having an exponent written, is considered as the first power of itself; that is, its exponent is 1.

EXERCISES

A. Write each of the following products without exponents. Name the exponent and the root of each power.

1. a^2	4. 2^3	7. a^2b^3	10. x^2y^3
2. b^3	5. m^5	8. x^4y	11. $a^2b^2c^2$
3. x^5	6. 8^2	9. xy^3	12. $3a^2b^4c^5$

B. In each of the following expressions, name the numerical coefficient and the exponent of each letter.

1. $3a^4$	4. $4m$	7. $5ab^2$	10. mnp^2
2. $5x^2$	5. y^3	8. y	11. $4a^4$
3. $3a^2b^3$	6. $3a^3$	9. $2ax^2y$	12. $6a^2b^4c^5d^6$

§ 28. Degree of a Polynomial. The ***degree of a term*** is the number of letters that are factors of it. The degree is found by adding the exponents of the literal factors. For example, $2ab^3$ is of the *fourth* degree.

The ***degree of a polynomial*** is that of the term of the highest degree in the polynomial. For example, $3ab+2ab^2+b^2$ is of the third degree, this being the degree of the highest term (the second term).

EXERCISES

1. Name the degree of each of the following monomials: a^2, ab, xy^2, $3ab$, $5ab^2$, $2a^2b^3$, $3ab^2c$, $4a$.

2. Name the degree of each of the following polynomials: $a^2+2ab+b^2$, $x+y$, a^2-a, $2a^2+3b^2$, a^2+2a+1, $a^2b+ab+a$, $4x+5$, x^2y+xy.

§ 29. Addition of Monomials. Terms that contain the same factor are ***similar*** terms or ***like*** terms. For example, $+5x$, $-3x$, and $+7x$ are similar, since each contains the factor x. Their sum is $+9x$.

In adding like terms, the coefficient of the common factor in the sum is found by adding its coefficients in the different terms.

EXERCISES

In each of the following exercises find the sum.

1. $-6a$, $+4a$ *Ans.* $-2a$.
2. $-13n$, $+7n$
3. $+8x^2$, $+3x^2$
4. $-3a$, $-2a$
5. $-5x$, $-2x$
6. $8m^2$, $-3m^2$
7. $-15x$, $-2x$
8. D, $-4D$
9. $-t$, $+t$
10. $-17a$, $+a$
11. $-5x$, $+6x$, $-2x$, $+x$, $-x$ *Ans.* $-x$.
12. $-4x^2$, $+3x^2$, $-2x^2$, $+8x^2$, $-10x^2$
13. $-8ab$, $-3ab$, $-4ab$, $+7ab$, $+ab$
14. $14t$, $-8t$, $-3t$, $-45t$
15. $68k$, $+34k$, $-16k$, $-3k$
16. $4s$, $-12s$, $+6s$, $+6s$, $-9s$
17. $20w^3$, $+30w^3$, $-60w^3$, $-10w^3$
18. $7b$, $+8b$, $-20b$, $+3b$, $-b$
19. $5x^2$, $-3x^2$, $-2x^2$, $-x^2$, $+3x^2$
20. $3ab$, $-7ab$, $-2ab$, $+ab$, $+2ab$, $-10ab$

§ 30. Addition of Polynomials.

EXAMPLE 1. Add: $3x-4$, $2x+1$, $x-7$.

SOLUTION. Arrange the polynomials so that the like terms are in the same column.

$$\begin{array}{r} 3x-4 \\ 2x+1 \\ x-7 \\ \hline 6x-10 \end{array}$$

Ans. $6x-10$.

EXAMPLE 2. Add: $3x-2y+3$, $-2x+3y$, $x-y-2$.

SOLUTION.

$$\begin{array}{rrr} 3x & -2y & +3 \\ -2x & +3y & \\ x & -\ y & -2 \\ \hline 2x & & +1 \end{array}$$

Ans. $2x+1$.

NOTE. A method of checking the addition of polynomials is by assigning number values to the letters.

EXERCISES

In each of the following exercises add the given polynomials.

1. $3x+2$, $4x-1$, $3x-12$, $2x+8$
2. $5a-3$, $2a+7$, $3a-5$, $4a+1$
3. $2y+7$, $y-4$, $y-2$, $3y+1$
4. $4b+3$, $-2b+4$, $-3b-12$, $2b+6$
5. $3h^2-7$, $2h^2+4$, $-2h^2+5$, $4h^2-2$
6. $3a-1$, $4a$, $-7a+3$, $2a-1$
7. $a-2$, $4+3a$, $7-2a$, $a-5$
8. $x+2y$, $2x-3y$, $x-7y$, $3x+y$
9. $2x^2-y^2$, $4y^2-2x^2$, x^2+7y^2, $5x^2-y^2$
10. a^2-2a, $2a-3a^2$, a^2+2a, $7a^2-a$

11. $c+d,\ c-2d,\ d-3c,\ c-d$

12. $4m,\ 2m-3p,\ p-4m,\ 5p$

13. $7a-2b,\ a-b,\ b-2a,\ 2b-7a$

14. $x^2-3x,\ x+2x^2,\ x-3x^2,\ -3x$

15. $3a,\ a-5b,\ b+3a,\ a-7b$

§ 31. Parentheses. The symbols [ ], called *brackets*, { }, called *braces*, and ( ), called *parentheses*, are used to group terms together. Thus the expression $(2a+5b)+(3a-2b)$ means that we are to add $2a+5b$ and $3a-2b$.

EXAMPLE 1. Simplify $(2a+5b)+(3a-2b)+(b-3a)$.

SOLUTION. $(2a+5b)+(3a\ \ -2b)\ +(b\ \ -3a)$

$$\begin{aligned}&2a+5b \quad +3a-2b \quad +b-3a && \text{(Parentheses removed)}\\ &2a+4b && \text{(Like terms collected)}\end{aligned}$$

Ans. $2a+4b$.

NOTE. The plus signs between the parentheses indicate that the quantities are to be *added*. Hence, in removing the parentheses, the signs of all the terms remain the *same*. This is shown by the arrows.

EXAMPLE 2. Simplify $(3x-2)+(-4-7x)+(3-x)$.

SOLUTION. $(3x-2)+(-4-7x)+(3\ -x)$,

$$\begin{aligned}&3x-2 \quad -4-7x \quad +3-x && \text{(Parentheses removed)}\\ &-5x-3 && \text{(Like terms collected)}\end{aligned}$$

Ans. $-5x-3$.

NOTE. In Algebra it is customary to add terms horizontally, as in examples 1 and 2, instead of arranging the expressions in columns.

EXERCISES

Simplify each of the following expressions.

1. $(a-2)+(a+3)+(2a-7)$
2. $(x+5)+(2x-3)+(x-2)$
3. $(2x^2-7)+(3x^2-1)+(2x^2-3)$
4. $(3a+2b)+(a-2b)+(b-3a)$
5. $(c-2d)+(-d-2c)+(c+3d)$
6. $(2x^2-x)+(3x^2-4x)+(x-7x^2)$
7. $(a-b+3)+(2a+b-1)$
8. $(3x-2y+1)+(-2-2y)+(3x-7)$
9. $(2a^2-3a+7)+(a-2a^2)+(a+4)$
10. $(x+y+1)+(x-y-1)+(y-2x)$
11. $(-3x^2-5)+(2x^2-7)+(-5x^2-3x+12)$
12. $(-5+8m^2+3m)+(-3m^2+4m)+(-7m-5m^2)$
13. $(15y^2-13)+(-7+6y-8y^2)+(-6y+7y^2)$
14. $(2.3x^2-3.4x+1.7)+(-2.4x^2+4.5x-2.1)$
15. $(-5.6m^2+3.9m-4.3)+(-3.4m^2-4.1m+5.4)$

§ 32. **Subtraction of Monomials.** ***Subtraction*** is the process of taking one number (called the ***subtrahend***) from another number (called the ***minuend***).

The process of subtraction is directly *opposite* to the process of addition. Subtracting a positive number is the same as adding a negative number. Subtracting a negative number is the same as adding a positive number.

From these statements the rule for subtracting one signed number from another is:

Change the sign of the subtrahend and combine as in the addition of signed numbers.

The following examples show all the different combinations of signed numbers for subtraction.

	Example	Subtrahend, with Sign Changed	Process, Add	Answer
1.	Take +4 from +7	−4	−4, +7	+3
2.	" +7 " +4	−7	−7, +4	−3
3.	" −4 " −7	+4	+4, −7	−3
4.	" −7 " −4	+7	+7, −4	+3
5.	" −4 " +7	+4	+4, +7	+11
6.	" +7 " −4	−7	−7, −4	−11
7.	" +4 " −7	−4	−4, −7	−11
8.	" −7 " +4	+7	+7, +4	+11

In subtraction, to check your answer, add it to the subtrahend; the result should be the minuend.

EXERCISES

A. In each of the following exercises, subtract the first number from the second.

1. +2, +7
2. +7, +3
3. −9, −15
4. −4, +8
5. +8, −3
6. +4, +1
7. +9, −12
8. −15, −1
9. −10, +3
10. −3, −12
11. −4, +4
12. −9, 12
13. +7, +7
14. −12, −18
15. −10, 5
16. −1, −1
17. 11, −18
18. 15, 8
19. −1, 9
20. $-2\frac{1}{2}$, $-5\frac{1}{2}$

Note. For additional drill in the subtraction of signed numbers, the Drill Table on page 69 may be used.

B. In each of the following exercises, take the second monomial from the first.

1. $+3x,\ -2x$	7. $4c,\ 7c$	13. $-2cd^2,\ -2cd^2$
2. $-3a,\ -2a$	8. $2d^2,\ -3d^2$	14. $15r,\ -20r$
3. $+3y,\ +2y$	9. $-5x,\ -5x$	15. $3t,\ -3t$
4. $-3b,\ +2b$	10. $y^3,\ -2y^3$	16. $x^2y,\ x^2y$
5. $-8y^2,\ -7y^2$	11. $7k,\ 7k$	17. $3b^3,\ -13b^3$
6. $+5m,\ -2m$	12. $15m,\ -m$	18. $-2\frac{1}{3}b,\ 4\frac{1}{2}b$

§ 33. Subtraction of Polynomials. *To subtract one polynomial from another, change the sign of* ***each*** *term of the subtrahend, and then add.*

EXAMPLE 1. Take $8x-3$ from $2x+7$, and check the result.

SOLUTION. Think the sign of each term of the subtrahend $(8x-3)$ changed, making it $-8x+3$; then add $-8x+3$ to $2x+7$, giving the sum, $-6x+10$.

The work appears as follows, the changing of signs being done mentally.

$$\begin{array}{rl} 2x+7 & \text{(Minuend)} \\ \underline{8x-3} & \text{(Subtrahend)} \\ -6x+10 & \text{(Remainder)} \end{array}$$

Check, by adding *upward* (Remainder+subtrahend= minuend). *Ans.* $-6x+10$.

EXAMPLE 2. From $2x-y+3$ take $x-3y-4$, and check the result.

SOLUTION. Here the subtrahend is $x-3y-4$.

$$\begin{array}{rl} 2x-\ \ y+3 & \text{(Minuend)} \\ \underline{x-3y-4} & \text{(Subtrahend)} \\ x+2y+7 & \text{(Remainder)} \end{array}$$

NOTE. As the sign of each term of the subtrahend $(x-3y-4)$ has been changed *mentally*, the change of signs is not shown in the work. Check by adding *upward*. *Ans.* $x+2y+7$.

EXAMPLE 3. From $12a-3b$ take $5a-5c$, and check the result.

SOLUTION. Here the subtrahend is $5a-5c$.

$$\begin{array}{l} 12a-3b \\ \underline{5a \qquad -5c} \\ 7a-3b+5c \end{array}$$

NOTE. Although there is no term containing c in the minuend, the sign of $-5c$ must be changed, because it is one of the terms of the subtrahend. Check by adding *upward*.

Ans. $7a-3b+5c$.

EXERCISES

In each of the following exercises find the remainder and check.

1. Take $3x+2$ from $4x-1$.
2. Subtract $5a-3$ from $3a-5$.
3. From $2y+7$ take $y-4$.
4. From $4b+3$ take $-2b+4$.
5. From $3h^2-7h$ take $4h^2-2h$.
6. Take $3a$ from $4a-1$. (See Example 3.)
7. Take $2x+5$ from $5x$. (See Example 3.)
8. From $x+2y-4$ take $x-3y-5$.
9. From $3a^2+2a-3$ take a^2+8.
10. From $5x-8$ take 15.
11. Subtract $a+2b$ from $-2a-7b+4$.
12. From $2a^2-7$ take $a-8$.
13. Take $2c+d$ from $d-7c$.
14. Subtract a^2-2b^2 from b^2+2a^2.
15. From x^3-2x^2+7x take $3x^3-x^2$.
16. Take $3x-y$ from $7x-12$.
17. From $x-y-7$ take $x-y+4$.

18. Take $3a-7$ from $3a-7$.

19. From $3x^2-5$ take y^2-12.

20. From $2a-5b$ take $3a-5b+4$.

21. Simplify $(4a-2)-(2a+4)$.

SOLUTION. $(4a-2)-(2a+4)$

$4a-2 \quad -2a-4$ (Parentheses removed)

$2a-6$ (Like terms collected)

Ans. $2a-6$.

NOTE. The minus sign between the parentheses indicates that the quantity following (the subtrahend) is to be subtracted. Hence, in removing the parentheses from the quantity to be subtracted, the signs of all its terms are changed. This is shown by the arrows.

22. Simplify $(3x-8)-(7x-2)$. *Ans.* $-4x-6$.

23. Simplify $(4b+1)-(7b-4)$.

24. Simplify $(a-2)+(a+3)-(2a-7)$.

25. Simplify $(x^2+3)-(2x^2-3)+(3x^2-4)$.

26. Simplify $(2x-7)+(-3-x)-(2x-8)$.

27. Simplify $(3a+2b)-(a-2b)-(-b+3a)$.

28. Simplify $(c^2-2d^2)+(-d^2-2c^2)-(c^2-3d^2)$.

29. Simplify $(2x-y)-(3x-2y)-(y-7x)$.

30. Simplify $(3a-b+2)-(2a+b-1)$.

31. Simplify $(2x^2-2x+1)+(2x-2)-(3x^2-7)$.

32. Simplify $(2a-b+3)-(b-3a)+(-6+b)$.

33. Simplify $(x^2+xy+y^2)-(-y^2+x^2-xy)+(y^2-3x^2)$.

34. Simplify $4-(x-y)+(2x+y+1)$.

35. Simplify $2a-(a+b)-3b+(a+5b)+a$.

CHAPTER VIII

MULTIPLICATION AND DIVISION OF ALGEBRAIC EXPRESSIONS

§ 34. Law of Signs in Multiplication

EXAMPLE 1. Find the product of $+4$ multiplied by $+3$.

SOLUTION. Consider $+3$ the multiplier, then

$(+3)\times(+4)=+12$, for this means

$(+4)+(+4)+(+4)$; that is, $+4$ is to be added three times. *Ans.* $+12$.

EXAMPLE 2. Find the product of -4 multiplied by $+3$.

SOLUTION. Consider $+3$ the multiplier, then

$(+3)\times(-4)=-12$, for this means

$(-4)+(-4)+(-4)$; that is, that -4 is to be added three times. *Ans.* -12.

EXAMPLE 3. Find the product of $+4$ multiplied by -3.

SOLUTION. Consider -3 the multiplier; as this multiplier is the negative of the multiplier in Example 1, this product is the negative of that product; that is,

$$(-3)\times(+4)=-12.$$

(Compare this answer with that of Example 1.)

Ans. -12.

EXAMPLE 4. Find the product of -4 multiplied by -3.

SOLUTION. Consider -3 the multiplier; as this multiplier is the negative of the multiplier in Example 2, this product is the negative of that product; that is,

$$(-3)\times(-4)=+12.$$

(Compare this answer with that of Example 2.) *Ans.* $+12$.

We may state these principles of signs as follows:
Plus times plus gives plus (Ex. 1).
Plus times minus gives minus (Ex. 2).
Minus times plus gives minus (Ex. 3).
Minus times minus gives plus (Ex. 4).

Summary.

These four principles may be further condensed into the statement of the following LAW OF SIGNS IN MULTIPLICATION:

Two like signs produce plus.
Two unlike signs produce minus.

§ 35. DRILL TABLE — COMBINATION OF SIGNS

A	+	−	−	+	+	−	+	−	+	−	+	−	−	−	+	−	+	−	−	+
B	+	−	+	+	−	+	−	−	−	+	+	−	+	−	+	−	+	−	+	−

SUGGESTION. Use the signs in line A as the signs of the multiplicand with those in line B as the signs of the multiplier. Find the sign of each product.

§ 36. Multiplication of Signed Numbers.

EXERCISES

Find each of the following indicated products:

1. $(+3)\times(-5)$
2. $(-5)\times(+4)$
3. $(-4)\times(-6)$
4. $(-8)\times(+9)$
5. $(+7)\times(-8)$
6. $(-9)\times(-9)$
7. $(+8)\times(-9)$
8. $(-7)\times(-9)$
9. $(-4)\times(+12)$
10. $(-6)\times(-11)$

In Exs. 11–20, find the product of the first two numbers, then multiply by the third number.

11. $(+2)\times(-3)\times(+4)$	**16.** $(-5)\times(-4)\times(-7)$
12. $(-3)\times(-2)\times(-6)$	**17.** $(+5)\times(-6)\times(-3)$
13. $(-4)\times(+3)\times(-4)$	**18.** $(+\frac{1}{2})\times(-6)\times(-4)$
14. $(-5)\times(-4)\times(+3)$	**19.** $(-\frac{3}{4})\times(+8)\times(-5)$
15. $(-2)\times(+8)\times(-3)$	**20.** $(-\frac{3}{4})\times(+12)\times(-\frac{2}{3})$

§ 37. DRILL TABLE — MULTIPLICATION OF SIGNED NUMBERS

A	$+1$	-2	$+3$	-5	-4	$+8$	-9	$+7$
B	-1	$+2$	$+5$	-2	-3	$+4$	$+5$	$+6$
C	-6	$+3$	$+5$	$+8$	$+9$	-8	-7	-4
D	$+\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$+\frac{3}{4}$	$+\frac{2}{3}$	0

Suggested drills with this table:

(*a*) Use line *A* as the multiplicand, and each number in line *B* as a multiplier.

(*b*) Use line *B* as the multiplicand, and each number in line *C* as a multiplier.

(*c*) Use each number in line *D* as a multiplier, and lines *A*, *B* and *C* as multiplicands.

NOTE. When zero is one of the factors, the product is zero.

(*d*) Find the product of the first three numbers in line *A*; the first four, etc.; similarly for the other lines.

(*e*) Find the product of all the numbers in each column.

NOTE. In finding the product of more than two signed numbers, the simplest way to determine the correct sign of the product is as follows: Count the number of minus signs of all the factors. If there is an *odd* number of minus signs the final product is *minus;* if there is an *even* number of minus signs, the final product is *plus*. For example, the final product of all numbers in line 1 is plus; in line 2, it is minus; etc.

§ 38. **Multiplication of Monomials.** In finding the product of two or more monomials, the factors may be written in any order, as in arithmetic. For example, just as $2 \times 3 \times 4 = 4 \times 2 \times 3$, so

$$3abc \equiv 3bac.$$

EXAMPLE 1. Find the product of $5a^2b^2$ and $-4ab^3$.

SOLUTION. The factors may be rearranged as follows:

$$5 \times (-4) \times a^2 \times a \times b^2 \times b^3 \equiv -20a^3b^5.$$

EXPLANATION. $(+5) \times (-4) = -20$ (Law of Signs.)

The total number of times that a appears as a factor is 3; 3 is obtained by adding the exponents of a, $(2+1)$.

The total number of times that b appears as a factor is 5; 5 is obtained by adding the exponents of b, $(2+3)$.

Ans. $-20a^3b^5$.

EXAMPLE 2. Find the product of $-3xy$, x^2y^2, and $-7x^2$.

SOLUTION. The product of the numerical coefficients (including signs) is $+21$. The product of the literal factors is x^5y^3.

$$(-3xy)(x^2y^2)(-7x^2) \equiv +21x^5y^3. \qquad \textit{Ans. } +21x^5y^3.$$

From the preceding examples you have observed that the exponent of any letter in the product is found by adding the exponents of that letter in the factors. This is the LAW OF EXPONENTS IN MULTIPLICATION.

Summary. *In finding the product of two or more monomials:*

(1) *The numerical coefficient is the product of the numerical coefficients in the given factors.*

(2) *The exponent of each letter in the product is the sum of the exponents of that letter in all of the given factors.*

(3) *The combined product is a monomial.*

EXERCISES

Find each of the following indicated products.

1. $3a^4b\times(-4ab^2)$
2. $-4a^2b^5\times 5a^2b^2$
3. $-3b^2x^4\times(-3b^3x^2)$
4. $-7c^3d^2\times(-2c^3d)$
5. $-a^2cd^3\times 2a^2c^2d^2$
6. $2cd^2\times(-5a^2cd^3)$
7. $a^2b\times 2ab^2\times(-3ab^2)$
8. $-2a^2x\times 3a^3xy^2\times(-2ay^2)$
9. $\frac{3}{4}c^2d\times(-8cd^3)$
10. $\frac{1}{2}c^2y\times\frac{1}{3}acy^2$
11. $\frac{2}{3}x^3y^2\times\frac{3}{4}x^2y^3$
12 $5a\times\dfrac{2a^2}{5}$
13. $-12x^2\times\dfrac{3xy^2}{24}$
14. $7a\times\dfrac{5a^2}{3}$
15. $\frac{2}{3}a\times\frac{1}{2}b$

§ 39. Multiplication of a Polynomial by a Monomial.

Example. Find the product of $2a^2b-3ab^2$ multiplied by $3a^2b^2$.

Solution. Consider $3a^2b^2$ the multiplier, then

$$3a^2b^2(2a^2b-3ab^2)\equiv 3a^2b^2(2a^2b)+3a^2b^2(-3ab^2)\equiv 6a^4b^3-9a^3b^4$$

Ans. $6a^4b^3-9a^3b^4$.

Figure 15 is a geometric illustration of the algebraic identity

$$a(b+c)\equiv ab+ac.$$

Figure 16 is a geometrical illustration of the algebraic identity

$$a(b-c)\equiv ab-ac.$$

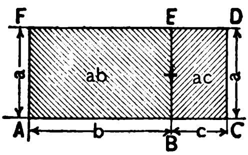

Fig. 15.

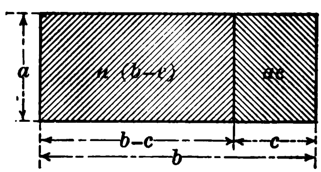

Fig. 16.

EXERCISES

Find each of the following indicated products.

1. $3(2a-5)$
2. $-2x(3x+1)$
3. $a^2(3a^2-2a-3)$
4. $2ab^2(3a-2bc)$
5. $3xy(2x^2-3xy+y^2)$
6. $-2mn(m-3n)$
7. $-3ab(2a^2-ab-3b^2)$
8. $5a^5b^2(7a^2-5ab+9b^2)$
9. $\frac{1}{2}a^3(4a-6b)$
10. $\frac{2}{3}x^2(6x^2-3xy+3y^2)$

§ 40. Multiplication of Polynomials.

EXAMPLE 1. Find the product of $2x+3$ and $3x-5$.

SOLUTION. The work may be arranged as in arithmetic.

$2x+3$	(Multiplicand)
$3x-5$	(Multiplier)
$6x^2+9x$	[$3x(2x+3)$, 1st partial product]
$-10x-15$	[$-5(2x+3)$, 2d partial product]
$6x^2-x-15$	(Product $\equiv$ sum of partial products)

Ans. $6x^2-x-15$.

EXAMPLE 2. Find the product of $2a^2-3a-1$ and $5a-2$.

SOLUTION.

$2a^2-3a-1$	(Multiplicand)
$5a-2$	(Multiplier)
$10a^3-15a^2-5a$	(1st partial product)
$-4a^2+6a+2$	(2d partial product)
$10a^3-19a^2+a+2$	(Product $\equiv$ sum of partial products)

Ans. $10a^3-19a^2+a+2$.

NOTE. A method of checking the multiplication of polynomials is by assigning number values to the letters.

EXERCISES

Find each of the following indicated products.

1. $(x+3)(x+2)$
2. $(x+5)(x-3)$
3. $(x-2)(x+4)$
4. $(x-5)(x-3)$
5. $(a-7)(2a-1)$
6. $(2a+3b)(3a-2b)$

7. $(3b-2a)(2b-3a)$
8. $(4a-1)(3a+5)$.
9. $(5x-2y)(3x+7y)$
10. $(7+3x)(5+2x)$
11. $(11+3x)(7-x)$
12. $(2x^2+3x+1)(3x-2)$
13. $(5a^2-4a+3)(2a-1)$
14. $(3m^2-5m+7)(2m-5)$
15. $(x^2+xy+y^2)(x-y)$
16. $(a^2-ab+b^2)(a+b)$
17. $(4+2x+x^2)(2-x)$
18. $(a+b)(a-b)$
19. $(a+b)(a+b)$
20. $(a-b)(a-b)$

21. $7\frac{2}{3}\times 3\frac{3}{4}$

SOLUTION.
$$\begin{aligned} 7\tfrac{2}{3}\times 3\tfrac{3}{4} &= (7+\tfrac{2}{3})(3+\tfrac{3}{4}) \\ &= 3(7+\tfrac{2}{3})+\tfrac{3}{4}(7+\tfrac{2}{3}) \\ &= 21+2+5\tfrac{1}{4}+\tfrac{1}{2} \\ &= 28\tfrac{3}{4} \end{aligned}$$
Ans. $28\frac{3}{4}$.

22. $6\frac{3}{4}\times 9\frac{2}{3}$
23. $8\frac{2}{3}\times 7\frac{3}{8}$
24. $9\frac{7}{8}\times 5\frac{5}{6}$
25. $7\frac{3}{4}\times 29\frac{2}{3}$
26. $13\frac{5}{8}\times 12\frac{1}{4}$
27. $30\frac{4}{5}\times 27\frac{3}{8}$
28. $75\frac{7}{8}\times 60\frac{2}{3}$
29. $83\frac{2}{3}\times 57\frac{1}{8}$
30. $95\frac{3}{8}\times 102\frac{4}{5}$

31. A room is reported to be 15 ft. by 13 ft. By a more accurate measurement it is found to be 15 ft. 3 in. by 13 ft. 4 in. Find, in square feet, the amount of the error in the area when it is computed from the first dimensions.

SOLUTION. $(15+\frac{1}{4})(13+\frac{1}{3})-15\times 13=8\frac{1}{3}$.

Ans. $8\frac{1}{3}$ sq. ft.

32. The dimensions of a room are reported as in Ex. 31. A more accurate measurement gives the dimensions as 14 ft. 9 in. by 12 ft. 10 in. Find, in square feet, the amount of the error in the area when it is computed from the first dimensions.

SOLUTION. $15\times 13-(15-\frac{1}{4})(13-\frac{1}{6})=5\frac{17}{24}$.

Ans. $5\frac{17}{24}$ sq. ft.

33. A room is reported to be 18 ft. by 12 ft. By a more accurate measurement it is found to be 17 ft. 9 in. by 11 ft. 10 in. Find in square feet the amount of error in the area when it is computed from the first dimensions.

34. The lighting area of a window pane is reported to be 10 in. by 8 in. A more accurate measurement gives it as $9\frac{7}{8}$ in. by $7\frac{3}{4}$ in. There are 12 panes of glass in this window. Find the total amount of error in the lighting area when it is computed from the first dimensions.

35. The dimensions of the floor of a room are given as 18 ft. 2 in. by 15 ft. 1 in. A less accurate measurement gives them as 18 ft. by 15 ft. Find in sq. ft. the difference in the floor area of this room when computed from each of these two sets of dimensions.

36. If the dimensions 18 ft. by 15 ft., in Ex. 35, are used in finding the area of the floor, what is the per cent error in the area, within a tenth of one per cent?

37. If you neglect the product term ab in $(1+a)(1+b)$, what is the per cent error, when $a=0.004$ and $b=0.005$, within a thousandth of one per cent?

38. The total lighting area of a room containing 5 windows, each with 12 panes of glass, each pane 12 in. by 10 in., is to be found. What will be the total amount of error in the lighting area as computed from these dimensions, if each dimension as given is $\frac{1}{4}$ in. too long? What is the per cent error, within a tenth of one per cent?

39. The dimensions of a room are taken as 28 ft. by 24 ft. The error in each dimension is not over $\frac{1}{4}$ ft. How great can the error in the area be?

SUGGESTION. $(28 \pm \frac{1}{4})(24 \pm \frac{1}{4}) - 28 \times 24 = ?$

§ 41. Special Products. Certain types of products occur so frequently in algebra that, like the Multiplication Table in arithmetic, they should be memorized.

The types of products which are most frequently met have been shown in exercises 18, 19, and 20, page 87. These are:

(1) $(a+b)(a-b)\equiv a^2-b^2.$

(2) $(a+b)(a+b)$, or $(a+b)^2\equiv a^2+2ab+b^2.$

(3) $(a-b)(a-b)$, or $(a-b)^2\equiv a^2-2ab+b^2.$

These products should be memorized. Each of these identities may be translated into words as follows:

(1) *The product of the sum and difference of two numbers is the difference of the squares of the two numbers.*

(2) *The square of the sum of two numbers is the square of the first, plus twice their product, plus the square of the second.*

(3) *The square of the difference of two numbers is the square of the first, minus twice their product, plus the square of the second.*

Figures 17–19 are the geometric illustrations of the algebraic identities in (1), (2), and (3).

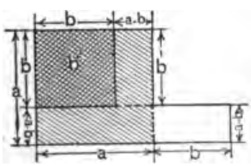

FIG. 17.

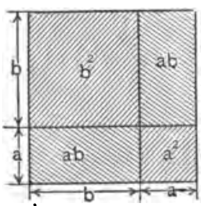

FIG. 18.

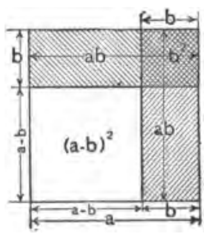

FIG. 19.

EXERCISES

Write each of the following indicated products by inspection.

1. $(a+c)(a-c)$	**11.** 31×29 [Write as: $(30+1)(30-1)$.]
2. $(x+4)(x-4)$	**12.** 51×49
3. $(a+c)^2$	**13.** 42×38
4. $(a-c)^2$	**14.** 59×61
5. $(x+y)^2$	**15.** 72×68
6. $(m+3)^2$	**16.** 51^2 [Write as: $(50+1)^2$.]
7. $(c-2)^2$	**17.** 31^2; 41^2; 91^2
8. $(y+5)(y-5)$	**18.** 42^2; 62^2
9. $(y+5)(y+5)$	**19.** 59^2 [Write as: $(60-1)^2$.]
10. $(y-5)(y-5)$	**20.** 49^2; 69^2

§ 42. Products of Two Binomials Having a Common Term.

EXAMPLE 1

$(x+5)(x+3)\equiv ?$

$$\begin{array}{r} x+5 \\ x+3 \\ \hline x^2+5x \\ +3x+15 \\ \hline x^2+8x+15 \end{array}$$

$x^2+8x+15$, or

$x^2+(5+3)x+15$

EXAMPLE 2

$(x-5)(x-3)\equiv ?$

$$\begin{array}{r} x-5 \\ x-3 \\ \hline x^2-5x \\ -3x+15 \\ \hline x^2-8x+15 \end{array}$$

$x^2-8x+15$, or

$x^2+(-5-3)x+15$

EXAMPLE 3

$(x+5)(x-3) \equiv ?$

$$\begin{array}{l} x+5 \\ x-3 \\ \hline x^2+5x \\ \quad -3x-15 \\ \hline x^2+2x-15, \text{ or} \\ x^2+(5-3)x-15 \end{array}$$

EXAMPLE 4

$(x-5)(x+3) \equiv ?$

$$\begin{array}{l} x-5 \\ x+3 \\ \hline x^2-5x \\ \quad +3x-15 \\ \hline x^2-2x-15, \text{ or} \\ x^2+(-5+3)x-15 \end{array}$$

From an examination of each product and a comparison of the four products (Examples 1–4), we observe the following:

(1) *That the first term of each product is x^2, the square of the common term,*

(2) *That the last term of each product is* ± 15, *the product of the other two terms,*

(3) *That the middle term of each product is the common term x with a coefficient which is the algebraic sum of the other two terms.*

Such products as examples 1–4 may be expressed in the general form,

$$(x+a)(x+b) \equiv x^2+(a+b)x+ab.$$

This identity may be translated into words as follows:

The product of two binomials having a common term is the square of the common term, plus the algebraic sum of the other two terms for the coefficient of the common term, plus the product of the other two terms. For example, $(x+7)(x-3) \equiv x^2+4x-21$; $(x-7)(x-3) \equiv x^2-10x+21$; etc.

Figure 20 is a geometric illustration of the algebraic identity

$$(x+a)(x+b) \equiv x^2+ax+bx+ab$$

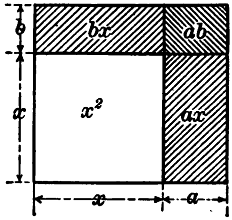

FIG. 20.

EXERCISES

Write each of the following indicated products by inspection.

1. $(x+4)(x+3)$
2. $(x-4)(x-3)$
3. $(x+4)(x-3)$
4. $(x-4)(x+3)$
5. $(y+6)(y-1)$
6. $(y-6)(y+1)$
7. $(y-6)(y-1)$
8. $(x+9)(x-5)$
9. $(x-9)(x-5)$
10. $(a-7)(a+2)$
11. $(a+7)(a+2)$
12. $(b+10)(b-3)$
13. $(b-10)(b-3)$
14. $(x-8)(x+7)$
15. $(x-7)(x+8)$
16. $(m-9)(m+10)$
17. $(c+2)(c-12)$
18. $(y+12)(y+3)$
19. $(y+12)(y-3)$
20. $(y-12)(y+3)$
21. $(y+9)(y+7)$
22. $(y-9)(y+7)$
23. $(x-8)(x-7)$
24. $(x-8)(x-9)$
25. $(2+x)(5+x)$
26. $(5+x)(2-x)$
27. $(4-x)(1+x)$
28. $(x+8)(x+8)$
29. $(x-7)(x-7)$
30. $(10+3)(10+2)$
31. $(10+5)(10+8)$
32. $(20+1)(20+2)$
33. $(20+2)(20+2)$
34. $(20+1)(20+5)$
35. $(20+3)(20+2)$
36. $(30+2)(30-1)$
37. $(100+2)(100+3)$
38. $(100-3)(100+2)$
39. $(1000+3)(1000-2)$
40. $(1000+5)(1000+2)$

§ 43. Products of Two Binomials Having Corresponding Terms Similar.

EXAMPLE 1

$(2x+1)(3x+2)\equiv ?$

$$\begin{array}{r} 2x+1 \\ 3x+2 \\ \hline 6x^2+3x \\ +4x+2 \\ \hline 6x^2+7x+2 \end{array}$$

EXAMPLE 2

$(2x-1)(3x-2)\equiv ?$

$$\begin{array}{r} 2x-1 \\ 3x-2 \\ \hline 6x^2-3x \\ -4x+2 \\ \hline 6x^2-7x+2 \end{array}$$

EXAMPLE 3

$(2x+1)(3x-2)\equiv ?$

$$\begin{array}{r} 2x+1 \\ 3x-2 \\ \hline 6x^2+3x \\ -4x-2 \\ \hline 6x^2-x-2 \end{array}$$

EXAMPLE 4

$(2x-1)(3x+2)\equiv ?$

$$\begin{array}{r} 2x-1 \\ 3x+2 \\ \hline 6x^2-3x \\ +4x-2 \\ \hline 6x^2+x-2 \end{array}$$

From an examination of each product and a comparison of the four products (Examples 1–4), we observe the following:

(1) *That the first and last terms of each product are the products of the corresponding similar terms of the binomials,*

(2) *That the middle term of each product is the algebraic sum of the cross-products of the dissimilar terms of the binomials.*

EXERCISES

Write each of the following indicated products by inspection.

1. $(2x+3)(3x+2)$
2. $(3x+1)(2x+3)$
3. $(2x-3)(x+2)$
4. $(3x-1)(2x+3)$
5. $(5a-2)(2a-1)$
6. $(3x-4)(2x-1)$

7. $(3y-5)(5y+2)$
8. $(4y-1)(y-6)$
9. $(3x-7)(x-2)$
10. $(3x+7)(x-2)$
11. $(2b+5)(5b-4)$
12. $(2b+3)(3b-2)$

§ 44. Law of Signs in Division.

EXAMPLE 1. Find the quotient of $+12$ divided by $+3$.

SOLUTION. $(+12)\div(+3)=+4$, since $(+3)\times(+4)=+12$. *Ans.* $+4$.

EXAMPLE 2. Find the quotient of -12 divided by $+3$.

SOLUTION. $(-12)\div(+3)=-4$, since $(+3)\times(-4)=-12$. *Ans.* -4.

EXAMPLE 3. Find the quotient of -12 divided by -3.

SOLUTION. $(-12)\div(-3)=+4$, since $(-3)\times(+4)=-12$. *Ans.* $+4$.

EXAMPLE 4. Find the quotient of $+12$ divided by -3.

SOLUTION. $(+12)\div(-3)=-4$, since $(-3)\times(-4)=+12$. *Ans.* -4.

We may condense these four principles of signs into the following LAW OF SIGNS IN DIVISION (See page 82.):

Two like signs produce plus.
Two unlike signs produce minus.

§ 45. Division of Signed Numbers.

EXERCISES

Find each of the following indicated quotients.

1. $+12\div(-6)$
2. $-32\div(-8)$
3. $+36\div(-4)$
4. $-48\div 6$
5. $-63\div(-7)$
6. $-54\div(-9)$

7. $-72 \div (+8)$
8. $-96 \div 8$
9. $-84 \div (-12)$
10. $81 \div (-9)$
11. $-100 \div (+4)$
12. $-120 \div (-8)$
13. $-132 \div 6$
14. $144 \div (-6)$
15. $-160 \div (-10)$
16. $-625 \div 25$
17. $-225 \div (-15)$
18. $+8 \div (-\frac{1}{2})$
19. $-24 \div \frac{2}{3}$
20. $-48 \div \frac{3}{4}$

§ 46. Division of Monomials.

EXAMPLE 1. Find the quotient of $-20a^3b^2 \div (-4ab)$.

SOLUTION. $(-20a^3b^2) \div (-4ab) \equiv +5a^2b$, since
$(+5a^2b) \times (-4ab) \equiv -20a^3b^2$.

Ans. $+5a^2b$.

EXAMPLE 2. Find the quotient of $(21x^5y^3) \div (-3xy)$.

SOLUTION. $(21x^5y^3) \div (-3xy) \equiv -7x^4y^2$, since
$(-7x^4y^2) \times (-3xy) \equiv +21x^5y^3$.

Ans. $-7x^4y^2$.

From the preceding examples you have observed that the exponent of any letter in the quotient is found by subtracting its exponent in the divisor from its exponent in the dividend. This is the LAW OF EXPONENTS IN DIVISION.

Summary. In finding the quotient of two monomials:

(1) *The numerical coefficient is the quotient of the numerical coefficient of the dividend divided by that of the divisor.*

(2) *The exponent of each letter in the quotient is its exponent in the dividend minus its exponent in the divisor.*

(3) *The combined quotient is a monomial.*

NOTE. Zero cannot be used as a divisor.

EXERCISES

Find each of the following indicated quotients.

1. $12a^3b^4 \div (-3ab^2)$
2. $-36x^5y^3 \div 6x^2y^3$
3. $-24a^5c^2 \div (-4a^2c)$
4. $-36a^6x^3 \div (-9a^2x)$
5. $-72m^4x^5 \div 8mx^2$
6. $63b^6y^4 \div 7b^2y^2$
7. $-96a^6m^4y^2 \div 12a^3m^2y$
8. $132n^6x^4 \div (-6n^2)$
9. $-240a^{12}b^8c^4 \div (-8a^4b^2c^4)$
10. $-1000x^{16}y^{12} \div 25x^8y^6$

§ 47. Division of Polynomials by Monomials.

EXAMPLE 1. Find the quotient of $(6x^4-2x^3+4x^2) \div 2x^2$, and check.

SOLUTION. The work may be arranged as in short division in arithmetic, each term of the dividend being divided by the divisor. The quotient is written above the dividend.

$$\begin{array}{rl} & \;\;3x^2-\;\;x\;+2 \quad \text{(Quotient)} \\ \text{(Divisor)}\;\; 2x^2 & \overline{)\,6x^4-2x^3+4x^2} \quad \text{(Dividend)} \end{array}$$

CHECK. This work may be checked by multiplying the *quotient* by the *divisor*. *Ans.* $3x^2-x+2$.

EXERCISES

Find each of the following indicated quotients.

1. $(24a^3-18a^2+12a) \div 6a$
2. $(16a^3b^3-12a^2b^2-8ab) \div 4ab$
3. $(32x^5-24x^4+20x^3) \div (-4x^3)$
4. $(54a^5b^3-27a^4b^4+18a^3b^5) \div 9a^3b^3$
5. $(63b^6c^4+49b^4c^6-35b^2c^8) \div 7b^2c^4$
6. $(7x^3y-21x^2y^2-14xy^3) \div 7xy$
7. $(6a^4-12a^3-6a^2) \div 6a^2$
8. $(a^2x^4-3x^3+bx^2) \div (-x^2)$

§ 48. Division of One Polynomial by Another Polynomial.

EXAMPLE 1. Divide x^2+5x+6 by $x+2$, and check.

SOLUTION. The work may be arranged as in long division in arithmetic. The principal steps are:

(1) Divide, (2) Multiply, (3) Subtract, (4) Bring down.

Check the work by multiplying the quotient by the divisor.

$$
\begin{array}{lrl}
 & x+3 & \text{(Quotient)} \\
\text{(Divisor)}\ x+2\big) & \overline{x^2+5x+6} & \text{(Dividend)} \\
 & x^2+2x & [x(x+2)] \\
\cline{2-2}
 & 3x+6 & \text{(Subtract, bring down)} \\
 & 3x+6 & [3(x+2)] \\
\cline{2-2}
 & 0 & \text{(Bring down)}
\end{array}
$$

CHECK.

$$
\begin{array}{r}
x+3 \\
x+2 \\
\hline
x^2+3x \\
2x+6 \\
\hline
x^2+5x+6
\end{array}
$$

Explanation of each step:

1ST TERM OF QUOTIENT, x	STEPS	2D TERM OF QUOTIENT, 3
$x^2 \div x$	Divide	$3x \div x$
$x(x+2)$	Multiply	$3(x+2)$
$(x^2+5x)-(x^2+2x)$	Subtract	$(3x+6)-(3x+6)$
6	Bring down	0

Ans. $x+3$.

H

Example 2. Divide $12x^2-23xy+5y^2$ by $3x-5y$, and check.

Solution.

$$\begin{array}{rl} & 4x-y \quad \text{(Quotient)} \\ \text{(Divisor)}\ 3x-5y) & \overline{12x^2-23xy+5y^2} \quad \text{(Dividend)} \\ & \underline{12x^2-20xy} \\ & \quad -3xy+5y^2 \\ & \quad \underline{-3xy+5y^2} \end{array}$$

Check.

$$\begin{array}{r} 4x-y \\ \underline{3x-5y} \\ 12x^2-3xy \qquad \\ \underline{-20xy+5y^2} \\ 12x^2-23xy+5y^2 \end{array}$$

Ans. $4x-y$.

Example 3. Divide x^3-5x^2+8x-6 by $x-3$, and check.

Solution.

$$\begin{array}{rl} & x^2-2x+2 \\ x-3) & \overline{x^3-5x^2+8x-6} \\ & \underline{x^3-3x^2} \\ & \quad -2x^2+8x \\ & \quad \underline{-2x^2+6x} \\ & \qquad\qquad 2x-6 \\ & \qquad\qquad \underline{2x-6} \end{array}$$

Check.

$$\begin{array}{r} x^2-2x+2 \qquad \\ \underline{x-3 \qquad\qquad} \\ x^3-2x^2+2x \qquad \\ \underline{-3x^2+6x-6} \\ x^3-5x^2+8x-6 \end{array}$$

Ans. x^2-2x+2.

EXERCISES

Find each of the following indicated quotients, and check each answer.

1. $(x^2-2x-8)\div(x+2)$
2. $(a^2+5a-14)\div(a-2)$
3. $(m^2+2m-35)\div(m+7)$
4. $(y^2-3y-54)\div(y-9)$
5. $(x^2-13x+40)\div(x-8)$
6. $(2a^2+7a-30)\div(a+6)$
7. $(6x^2+7x-5)\div(3x+5)$
8. $(6b^2+17b-14)\div(3b-2)$
9. $(15a^2-51ab+18b^2)\div(5a-2b)$
10. $(4x^2-27xy+44y^2)\div(x-4y)$
11. $(x^2-16)\div(x+4)$
12. $(a^2-4b^2)\div(a-2b)$
13. $(x^2-y^2)\div(x-y)$
14. $(16a^2-b^2)\div(4a-b)$
15. $(64-9x^2)\div(8+3x)$
16. $(x^3-2x^2+4x-8)\div(x-2)$
17. $(y^3+4y^2+y-6)\div(y+3)$
18. $(a^3-15a^2+65a-63)\div(a-7)$
19. $(3x^3+28x^2+29x-140)\div(3x-5)$
20. $(x^4-8x^3+10x^2+32x-35)\div(x-5)$
21. $(x^3+1)\div(x+1)$
22. $(a^3-b^3)\div(a-b)$
23. $(x^3+y^3)\div(x+y)$
24. $(y^3-8)\div(y-2)$
25. $(27x^3-8y^3)\div(3x-2y)$

26. $(x^2-x-6)\div(x-2)$ *Ans.* $x+1-\dfrac{4}{x-2}$.

27. $(2x^2+5x-6)\div(2x-5)$

28. $(x^3+5x^2-3x+8)\div(x-3)$

29. $(a^3+b^3)\div(a-b)$

30. $(x^4+1)\div(x+1)$

§ 49. Equations Involving Parentheses.

EXAMPLE 1. Solve the equation $3(2x-2)=2(x+3)$, and check the answer.

SOLUTION.

①	$3(2x-2)=2(x+3)$	
②	$6x-6=2x+6$	①≡(Parentheses removed)
③	$6x-2x=6+6$	②$+6-2x$
④	$4x=12$	③≡(Like terms collected)
⑤	$x=3$	④$\div 4$

CHECK.

$$3(2\times3-2)\stackrel{?}{=}2(3+3)$$
$$3(6-2)\stackrel{?}{=}2(6)$$
$$12=12$$

Ans. $x=3$.

NOTE. In evaluating expressions which involve the four fundamental operations, multiplication and division should be performed before addition and subtraction. If parentheses are involved, the expressions within the parentheses should be evaluated first.

EXAMPLE 2. Solve the equation $2(5x-7)-3(2x-9)=15$, and check the answer.

SOLUTION.

①	$2(5x-7)-3(2x-9)=15$	
②	$(10x-14)-(6x-27)=15$	①≡(Multiplying)
③	$10x-14-6x+27=15$	②≡(Parentheses removed)

④ $4x+13=15$ ③ ≡ (Like terms collected)
⑤ $4x=2$ ④ -13
⑥ $x=\frac{1}{2}$ ⑤ $\div 4$

CHECK.

$$2(5\times\tfrac{1}{2}-7)-3(2\times\tfrac{1}{2}-9)\stackrel{?}{=}15$$
$$2(2\tfrac{1}{2}-7)-3(1-9)\stackrel{?}{=}15$$
$$2(-4\tfrac{1}{2})-3(-8)\stackrel{?}{=}15$$
$$-9+24=15$$

Ans. $x=\frac{1}{2}$.

EXAMPLE 3. Solve the equation $(x+3)(2x-1)-4=(2x+5)(x-7)$, and check the answer.

SOLUTION.

① $(x+3)(2x-1)-4=(2x+5)(x-7)$
② $(2x^2+5x-3)-4=(2x^2-9x-35)$ ① ≡ (Multiplying)
③ $2x^2+5x-7=2x^2-9x-35$ ② ≡ (Parentheses removed)
④ $+14x=-28$ ③ $-2x^2+9x+7$
⑤ $x=-2$ ④ $\div 14$

CHECK. $$(-2+3)[2(-2)-1]-4\stackrel{?}{=}[2(-2)+5](-2-7)$$
$$(+1)(-4-1)-4\stackrel{?}{=}(-4+5)(-9)$$
$$(+1)(-5)-4\stackrel{?}{=}(+1)(-9)$$
$$-9=-9$$

Ans. $x=-2$.

EXERCISES

Solve each of the following equations, and check each answer.

1. $5(3x+1)-7x=3(x-7)+31$
2. $3+2(x-3)=3(x-3)$
3. $5(x-2)-7x=3(2x-3)-7$
4. $3+2(x-6)=3(2-x)$

5. $14x+3(3+x)=2(3x-1)$

6. $12x-2(4x-7)=18$

7. $7x-12-2(x-5)=x+22$

8. $9y-3(2y-4)=8-(6-y)$

9. $4-3(1-x)=3-(3x-1)$

10. $2(x-2)-3(6-x)=9(3+2x)+3$

11. $(x-2)(x-5)=(x+3)(x+2)$

12. $(x-5)(x-3)=(x-8)(x+2)$

13. $(2x+5)(4x+7)=8x(x+3)$

14. $4y(6y-1)+27=8y(3y+2)+147$

15. $(3x-5)(4x+3)-1=(4x-1)(3x+7)$

16. $12-3x(8x+5)=21-4x(6x-3)$

17. $13x-(x-5)(x+7)=37-(x-6)(x+2)$

18. $2(3x-1)(2x+5)-(4x-7)(3x+2)-238=0$

19. $8\left(\frac{x}{2}-\frac{3}{4}\right)-6\left(\frac{x}{3}-\frac{5}{3}\right)=4\frac{1}{2}$

20. $3(2.7x-.8)-1.2(5x-3)=11.7$

21. $\frac{4x}{3}+\frac{2x}{5}=\frac{26}{5}$

22. $\frac{n}{2}+\frac{n}{3}-\frac{n}{4}+\frac{n}{10}=82$

23. $\frac{3}{4}x+1\frac{1}{2}=\frac{7}{8}x$

24. $\frac{9y+9}{5}=y+1$

25. $\frac{b+12}{2}=b+5$

26. $\frac{3x-1}{4}+\frac{2x-3}{2}=5\frac{1}{4}$

27. $\frac{7x-5}{2}-\frac{3x+1}{6}=\frac{6x+1}{3}$

$[3(7x-5)-(3x+1)=2(6x+1);\ x=3.]$

28. $\frac{9x+3}{2}-\frac{2x-5}{3}=3x+4$

29. $\frac{x+5}{4}-\frac{3x-4}{5}=\frac{3}{4}-x$

30. $\frac{4x-1}{3}-\frac{2x+3}{2}=\frac{x-1}{2}-\frac{3x+7}{6}$

PROBLEMS

1. The length of a rectangle exceeds its width by 3 in. and its area exceeds the area of a square constructed on its width by 18 sq. in. Find the dimensions of the rectangle.

2. The length of a rectangle exceeds its width by 3 in. and its area is 36 sq. in. less than the area of a square constructed on its length. Find the dimensions of the rectangle.

3. Hazel is 25 years old and her brother Seth is 14 years old. How many years ago was Hazel twice as old as Seth?

4. John's age is two thirds of his father's age. How many years ago was John one fourth as old as his father, if John is now 42 years old?

5. My oldest brother is 7 years older than myself. Thirty-three years ago he was just twice as old. Find our ages now.

6. Three years ago John's age was twice that of his brother Charles. Three years hence Charles' age will be $\frac{5}{8}$ John's age. Find their ages now.

7. The sum of the fourth, eighth, and sixteenth parts of a certain number is 49. Find the number.

8. There is a certain number from $\frac{1}{3}$ of which if you take $\frac{1}{4}$ of it, you get the number 7. Find the number.

9. Find that number which when it is divided by 15 will give a quotient of 13 and a remainder of 13.

10. Separate the number 120 into three parts such that the second part shall be 2 more than the first part, and that the third part shall be 3 times the sum of the first and second parts.

11. A motorist rode 120 miles in $4\frac{1}{2}$ hours. Part of the distance traveled was in the country at the average rate of 30 miles per hour, and the rest within city limits at the average rate of 10 miles per hour. For how many hours was he riding in the country?

12. A steamship left port at the average rate of 15 knots per hour. When it was at a certain distance from port, it became disabled and returned at the average rate of 4 knots per hour. It left port at 11.30 A.M. and had returned at 2.40 P.M. How far from port was the steamship when the accident happened? What was its average rate for the entire trip?

CHAPTER IX

PAIRS OF LINEAR EQUATIONS

§ 50. Graphs of Linear Pairs. In Chapter VIII, Second Course, pages 196–200, the graphs of linear equations were plotted on squared paper. To definitely locate a straight line on squared paper two points were necessary, and a third point was obtained to check the location of the two points, hence it was necessary first to tabulate three pairs of values. To get these values, it was found convenient to solve the equation for x or for y.

Example 1. Plot the graph of the equation $x+2y=8$. Using the same axes, plot the graph of the equation $2x-y=6$. From the graphs, find the pair of values that satisfies both equations.

Solution. (I) $x+2y=8$ Solving for x, $x=8-2y$
When $y=4$, $x=0$. When $y=3$, $x=2$. When $y=1$, $x=6$.

x	0	2	6
y	4	3	1

(II) $2x-y=6$

Solving for x, $x=\dfrac{6+y}{2}$

x	3	5	7
y	0	4	8

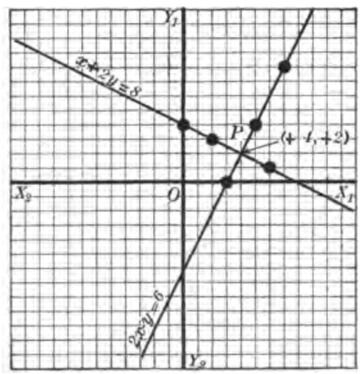

Fig. 21.

The graphs of these equations intersect at $(+4, +2)$; that is, $x=4$, $y=2$ is the pair of values that satisfies both equations.

CHECK. (I) $4+4=8$

(II) $8-2=6$

Ans. $x=4$, $y=2$.

These simultaneous equations are called ***independent equations*** because they have distinct graphs.

X_1X_2 and Y_1Y_2 are called *coördinate axes*. $(+4, +2)$ are *coördinates* of the point P. O is called the *origin*.

EXAMPLE 2. Plot the graphs of the pair of equations $2x+y=7$ and $4x+2y=8$.

SOLUTION. (I) $2x+y=7$

Solving for y, $y=7-2x$

x	0	2	4
y	7	3	-1

(II) $4x+2y=8$

Solving for y, $y=4-2x$

x	0	2	4
y	4	0	-4

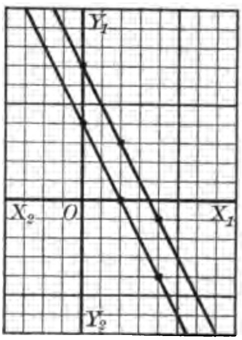

FIG. 22.

These graphs are parallel, hence no pair of values satisfies both equations. These simultaneous equations are called ***inconsistent equations***.

EXERCISES

Plot the graphs for each of the following pairs of equations. In exercises where the two equations are independent, prove that the pair of values for the point of intersection of the two graphs satisfies both of the equations.

1. $\begin{cases} x+y=6 \\ x+2y=8 \end{cases}$

2. $\begin{cases} 2x+5y=10 \\ 4x-2y=8 \end{cases}$

3. $\begin{cases} 2x+5y=15 \\ x-y=4 \end{cases}$

4. $\begin{cases} x-2y=5 \\ 3x-6y=9 \end{cases}$

5. $\begin{cases} x+4y=9 \\ 3x-y=14 \end{cases}$

6. $\begin{cases} 3x+4y=10 \\ 5x-y=9 \end{cases}$

§ 51. Solution of Linear Pairs by Addition or Subtraction. When linear pairs are independent, they can be solved by *eliminating* or getting rid of, one unknown, thereby forming one equation. The process of elimination that we shall use here is by *addition* or *subtraction*.

EXAMPLE 1. Solve the pair of equations:

(I) $$5x+3y=8$$

(II) $$4x+5y=-4$$

SOLUTION. To eliminate y multiply (I) by 5 and (II) by 3, then subtract the resulting equations, because the signs of the terms containing y are *alike*.

③	$25x+15y=40$	(I)×5
④	$12x+15y=-12$	(II)×3
⑤	$13x=52$	③−④
⑥	$x=4$	⑤÷13

Substituting 4 for x in (I),

⑦	$20+3y=8$	(I)≡
⑧	$3y=-12$	⑦-20
⑨	$y=-4$	⑧$\div 3$

CHECK. (I) $20-12=8$

(II) $16-20=-4$

Ans. $x=4$, $y=-4$.

EXAMPLE 2. Solve the pair of equations:

(I) $$\frac{3x+5}{2}+y=14$$

(II) $$\frac{6x}{3}-\frac{y+8}{2}=4$$

SOLUTION. First get the equations in the form of those in Ex. 1.

③	$3x+5+2y=28$	(I)$\times 2$
④	$3x+2y=23$	③-5
⑤	$12x-3y-24=24$	(II)$\times 6$
⑥	$12x-3y=48$	⑤$+24$
⑦	$12x+8y=92$	④$\times 4$
⑧	$11y=44$	⑦$-$⑥
⑨	$y=4$	⑧$\div 11$

Substituting 4 for y in ④, $x=5$

CHECK. (I) $\frac{3\times 5+5}{2}+4\stackrel{?}{=}14$

$10+4=14$

(II) $\frac{6\times 5}{3}-\frac{4+8}{2}\stackrel{?}{=}4$

$10-6=4$ *Ans.* $x=5$, $y=4$.

EXERCISES

Solve the following pairs of equations and check the answers.

1. $\begin{cases} 5x+2y=8 \\ 3x+2y=4 \end{cases}$

2. $\begin{cases} a+b=7 \\ 2a+3b=17 \end{cases}$

3. $\begin{cases} 5m+6n=16 \\ 3m+4n=10 \end{cases}$

4. $\begin{cases} 5p+2w=22 \\ 3p-4w=8 \end{cases}$

5. $\begin{cases} 4a+9b=5 \\ 2a+3b=2 \end{cases}$

6. $\begin{cases} 6a+3b=12 \\ 4a-5b=8 \end{cases}$

7. $\begin{cases} 3x+y=-2 \\ 4x-y=12 \end{cases}$

8. $\begin{cases} 5m+4n=2 \\ 2m-n=-7 \end{cases}$

Transform the equations in Exs. 9–16, so that they will be in the form of those in Exs. 1–8, before eliminating.

9. $\begin{cases} 5x=1-4y \\ 4y=15-3x \end{cases}$

10. $\begin{cases} \dfrac{7a-15}{3}=b \\ 5a-3b=9 \end{cases}$

11. $\begin{cases} \dfrac{a+b}{2}+\dfrac{a-b}{2}=20 \\ a-b=10 \end{cases}$

12. $\begin{cases} \dfrac{x}{8}-\dfrac{y}{9}=0 \\ \dfrac{x}{3}-\dfrac{y}{4}=\dfrac{5}{12} \end{cases}$

13. $\begin{cases} \dfrac{a}{3}-\dfrac{y}{4}=\dfrac{1}{12} \\ a-\dfrac{3y}{8}=1 \end{cases}$

14. $\begin{cases} \dfrac{x+3}{2}=9-5y \\ \dfrac{y+9}{10}=\dfrac{x-2}{3} \end{cases}$

15. $\begin{cases} \dfrac{m+2}{3}-\dfrac{n-5}{3}=0 \\ \dfrac{2m-7}{3}=\dfrac{13-n}{6}+10 \end{cases}$

16. $\begin{cases} \dfrac{x+5}{6}-\dfrac{y+4}{5}=0 \\ \dfrac{x-3}{2}=5-\dfrac{y+6}{4} \end{cases}$

PROBLEMS

1. A farmer paid 10 men and 8 boys $51 for a day's work. Later he paid 4 men and 6 boys $26 for a day's work. How much did he pay each man per day? How much did he pay each boy?

2. For an entertainment, tickets were sold for 35 cents and 25 cents. The total proceeds were $100 for 320 tickets. How many of each kind were sold?

3. For the same entertainment the next night the total proceeds were $140.10 for 450 tickets. How many of each kind were sold?

4. A grocer has two kinds of tea, one selling at 40 cents per pound and the other at 50 cents. How many pounds of each must be used to make a mixture of 10 pounds to sell for $4.40?

5. Milk is sold at 12 cents per quart and heavy cream at 60 cents per quart. How many quarts of each will be needed to make 18 quarts of light cream to sell for $6?

6. A grocer has two kinds of coffee, one selling at 22 cents per pound and the other at 32 cents. How many pounds of each kind must be used to make 12 pounds to sell for $3.20?

7. A mixture of 7-cent rice and 11-cent rice is to be sold at 3 pounds for a quarter. How many pounds of each must be used to make up sixty 3-pound packages?

8. Nougatines selling at 40 cents per pound are to be mixed with chocolate almonds selling at 60 cents per pound to make a mixture to sell at 48 cents per pound. If 15 pounds of the mixture are wanted, how many pounds of each must be used?

9. Forty laborers were engaged to raze a building. Some of them agreed to work for \$2.25 per day and others for \$2.50. The total amount paid them per day was \$92. How many worked at each rate?

10. A collection of nickels and dimes, containing 46 coins, amounted to \$2.90. How many coins of each kind were there?

11. A part of \$1200 is invested at 6% and the remainder at 5%. The combined yearly income is \$68.50. Find the number of dollars in each investment.

12. A man invested \$750 in Liberty Bonds, part of it in $3\frac{1}{2}$% and the rest of it in $4\frac{1}{2}$% bonds. His annual income is \$29.25. How many dollars were invested in each kind?

13. In playing teeter, two boys use a board 16 feet long. One boy weighs 90 pounds and the other weighs 110 pounds. At what point must the board be supported to balance?

14. A man rows 12 miles down stream in 3 hours and returns in 4 hours. Find the rate of the river and his rate of rowing in still water.

SOLUTION. Let m = the number of miles per hour that the man rows in still water, and
r = the number of miles per hour that the river flows, then
$m+r$ = the number of miles per hour that the man rows down stream, and
$m-r$ = the number of miles per hour that the man rows up stream.

The equations are:

(I) $3(m+r)=12$

(II) $4(m-r)=12$

Solving, $m=3\frac{1}{2}$, $r=\frac{1}{2}$. *Ans.* $m=3\frac{1}{2}$, $r=\frac{1}{2}$.

15. A boat goes down stream 30 kilometers in 3 hours and up stream 24 kilometers in 3 hours. Find its rate in still water and the rate of the current.

16. A man rows in still water at the rate of 3 miles per hour. It takes him 2 hours to go down stream to the next village. He returns in 4 hours. How far away is the village and what is the rate of the river?

17. A man rows in still water at the rate of 7 kilometers per hour. It takes him 30 minutes to row down stream to a certain island, and 1 hour and 15 minutes to return. How far away is the island and what is the rate of the current?

18. An airplane travels with the wind from one city to another 270 miles away in $2\frac{1}{4}$ hours and returns in $4\frac{1}{2}$ hours against a wind of the same velocity. Find the velocity of the wind and the rate of the airplane, if no wind is blowing.

19. A camping party sends a boy with mail to the nearest post office at 6 A.M. At 6.45 A.M. another boy is sent to overtake the first, which he does in $1\frac{1}{2}$ hours. If the second boy travels $1\frac{1}{2}$ miles per hour faster than the first boy, what is the rate of each?

CHAPTER X

FACTORS AND EQUATIONS

§ 52. **Factors.** If an expression is the product of two or more numbers, then these numbers are ***factors*** of the expression. Thus, pairs of factors of 30 are 2 and 15, 3 and 10, or 6 and 5. The ***prime*** factors of 30 are 2, 3, and 5, since each factor is a prime number. (A prime number is a number that is exactly divisible only by itself and by one.)

The factors of $6ab-15ac$ are $3a(2b-5c)$, since,

$$3a(2b-5c)\equiv 6ab-15ac$$

The factors of x^2-9y^2 are $x+3y$ and $x-3y$, since

$$(x+3y)(x-3y)\equiv x^2-9y^2$$

The factors of $a^2-8a+16$ are $a-4$ and $a-4$, since

$$(a-4)^2\equiv a^2-8a+16$$

§ 53. **Monomial Factors in Polynomials.**

TYPE I. $am+bm-cm$.

$$am+bm-cm\equiv m(a+b+c)$$

In this polynomial m represents any monomial that is a factor of each term. In factoring an expression of this type, m should be the *greatest common factor* of all the terms; that is, it should contain every factor that appears in *all* the terms of the polynomial.

EXAMPLE 1. Factor $5x^2y-10xy^2$.

$$5x^2y-10xy^2\equiv 5xy(x-2y)$$

CHECK. Multiply $x-2y$ by $5xy$.

Why is the symbol $\equiv$ used? (See page 50.)

EXAMPLE 2. Factor $12a^3-8a^2+4a$.

$$12a^3-8a^2+4a\equiv 4a(3a^2-2a+1)$$

CHECK. Multiply $3a^2-2a+1$ by $4a$.

Why do you take out the common factor $4a$ rather than a?

EXERCISES

Factor each of the following polynomials and check.

1. $3x+6$
2. $8a^2-3a$
3. $ab+b^2$
4. $5x+5$
5. c^3-3c^2-c
6. $a^2+2ab+a$
7. $3y^2-15y+9y^3$
8. $5x^2+10xy+15y^2$
9. $5x^3-10x^5$
10. $4a^2x^2-6a^2y^2+4a^2z^2$
11. $6x^2-x$
12. $16a^2-12a^3$
13. $28ax-49bx$
14. $2xy+8xy^2-12x^2$
15. $16x^2y-12xy^2+4xy$
16. x^2-x
17. $15a^2y-10a^2y^2+3a^2y^3$
18. $2m^2n-2mn^2$
19. a^3-a
20. $5a^2-10a-5$

§ 54. Binomials — The Difference of Two Squares.

TYPE II. a^2-b^2.

$$a^2-b^2\equiv(a+b)(a-b)$$

In this type the expression consists of two squares connected by a minus sign.

Since $(a+b)(a-b)\equiv a^2-b^2$ (See page 89.), it follows that the factors of a^2-b^2 are $a+b$ and $a-b$.

EXAMPLE 1. Factor m^2-9.

m^2 is the square of m, 9 is the square of 3, and the squares are connected by a minus sign; hence,

$$m^2-9\equiv(m+3)(m-3)$$

CHECK. Multiply $m+3$ by $m-3$.

EXAMPLE 2. Factor $1-25m^4$.

$$1-25m^4 \equiv (1+5m^2)(1-5m^2)$$

CHECK. Multiply $1+5m^2$ by $1-5m^2$.

Could the factor $1-5m^2$ be written first? Prove it.

EXERCISES

Factor each of the following binomials and check.

1. x^2-y^2
2. m^2-n^2
3. $25-x^2$
4. a^2-1
5. $100-a^2$
6. $16x^2-9y^2$
7. m^2-9x^4
8. $49a^2-9m^2$
9. D^2-4d^2
10. $1-b^2$
11. a^2b^2-4
12. $1-x^2y^2$
13. h^2-16b^2
14. $9a^2-25b^4$
15. $25a^2b^2-x^4$
16. $144y^2-1$
17. $9a^2b^2-49c^2$
18. $d_1{}^2-d_2{}^2$
19. $100m^4-1$
20. $R^2-4r_1{}^2r_2{}^2$
21. $9x^2+4$

Is $9x^2+4$ factorable under the type a^2-b^2?

In what respect does it differ from the type?

22. $4x^4-8$

Is $4x^4-8$ factorable under the type a^2-b^2?

In what respect does it differ from the type?

If it were $4x^2-9$, what would be its factors?

In Exs. 23–32, some of the given binomials cannot be factored as the difference of two squares, while others can be. State in what respect each of the former differs from the type.

23. $4a^2-1$
24. c^3-a^2
25. a^2+4
26. $4x^3-9y^2$

27. $9a^2-20b^2$

28. $49c^2-16d^2$

29. $4x^3-1$

30. b^2+x^2

31. $100-x^2y^2$

32. x^2y^3-100

33. $2x^2-18y^2$

If the monomial factor 2 is taken out, the other factor is of the type a^2-b^2, hence

$$2x^2-18y^2\equiv 2(x^2-9y^2)\equiv 2(x+3y)(x-3y)$$

In Exs. 34–43, find the *prime* factors by removing the monomial factor first.

34. ax^2-ay^2

35. $5a^2-20$

36. x^2y-y^3

37. $8a^3-2ab^2$

38. $4x^2-36$

39. $9x^3-4x$

40. $25x^3-x$

41. a^3-a

42. $2y^2-2$

43. $x^2y^3-4yz^2$

44. a^4-y^4

Solution.

$$a^4-y^4\equiv(a^2+y^2)(a^2-y^2)\equiv(a^2+y^2)(a+y)(a-y)$$

In Exs. 45–52, find the prime factors by factoring any factor that is not prime.

45. y^4-16

46. x^4-81

47. y^4-1

48. $1-x^4y^4$

49. a^4-16

50. ax^4-16a

51. x^8-y^8

52. x^5-x

53. $(a-b)^2-c^2$

Solution. $(a-b)^2-c^2\equiv(a-b+c)(a-b-c)$

54. $(2a+3)^2-16$

Solution. $(2a+3)^2-16\equiv(2a+3+4)(2a+3-4)$
$\equiv(2a+7)(2a-1)$

In Exs. 55–64, factor each expression as the difference of two squares and collect terms in the factors when possible.

55. $(m-n)^2-x^2$	**60.** $(2a+3)^2-4$
56. $(2a+b)^2-c^2$	**61.** $(5x-2)^2-9$
57. $(x-2y)^2-9$	**62.** $(3b+7)^2-25$
58. $(3a-b)^2-25$	**63.** $(4b^2-5)^2-16$
59. $(m^2-3m)^2-36$	**64.** $(9x^2-10)^2-36$

§ 55. Trinomials — The Squares of Binomials.

TYPE III. $a^2 \pm 2ab + b^2$.

$$a^2+2ab+b^2 \equiv (a+b)^2$$
$$a^2-2ab+b^2 \equiv (a-b)^2$$

A trinomial square contains two terms that are positive squares and a third term that is twice the product of their square roots. (See page 89.)

EXERCISES

In each of the following expressions, supply the missing term that will make it a trinomial square.

1. $a^2+(\quad)+b^2$	**11.** $(\quad)-12x+36$
2. $m^2-(\quad)+1$	**12.** $25a^2+(\quad)+9$
3. $x^2-10xy+(\quad)$	**13.** $9x^4-6ax^2y^2+(\quad)$
4. $4a^2+4ax+(\quad)$	**14.** $a^2+b^2+(\quad)$
5. $(\quad)-6a+9$	**15.** $4x^2+9y^2+(\quad)$
6. $(\quad)-30xy+25y^2$	**16.** $25+a^2+(\quad)$
7. $4x^2-8xyz+(\quad)$	**17.** $m^2+144-(\quad)$
8. $16x^2-16x+(\quad)$	**18.** $a^2+81-(\quad)$
9. $9-12a+(\quad)$	**19.** $x^2+64+(\quad)$
10. $64m^2+(\quad)+9$	**20.** $121+x^2+(\quad)$

§ 56. Factors of Trinomial Squares.

EXAMPLE 1. Factor $x^2+6xy+9y^2$.

$$x^2+6xy+9y^2 \equiv (x+3y)^2$$

CHECK. Multiply $x+3y$ by $x+3y$.

EXAMPLE 2. Factor $16x^4-8x^2y+y^2$.

$$16x^4-8x^2y+y^2 \equiv (4x^2-y)^2$$

CHECK. Multiply $4x^2-y$ by $4x^2-y$.

EXERCISES

Factor each of the following trinomials and check.

1. $x^2-2xy+y^2$
2. $m^2+2mn+n^2$
3. $16a^2-40ab+25b^2$
4. $4m^2+4mn+n^2$
5. a^2+x^2+2ax
6. b^2+m^2-2bm
7. $100+s^2+20s$
8. $9x^2+y^2-6xy$
9. $x^4+2x^2y^2+y^4$
10. $a^4-2a^2b+b^2$
11. a^2+4b^2-4ab
12. $64+m^2-16m$
13. $9a^2-30ab+25b^2$
14. $16x^2+40xy^2+25y^4$
15. $16+x^2-8x$
16. $1+a^2-2a$
17. x^4+2x^2+1
18. $4a^2-4ab^2+b^4$
19. m^2+1-2m
20. $49a^4b^4+1-14a^2b^2$
21. $4a^2+2a+1$

Is $4a^2+2a+1$ factorable under the type $a^2+2ab+b^2$? In what respect does it differ from the type?

In Exs. 22–31, some of the given trinomials cannot be factored as squares of binomials, while others can be. State in what respect each of the former differs from the type.

22. $a^2+4ab+4b^2$
23. $16m^2+4m+1$
24. $x^2-6xy+9y^2$
25. $9x^2+6xy+4y^2$

26. $x^2+2x^2y+y^2$

27. $64+x^2-16x$

28. $16x^2+50xy+25y^2$

29. $a^2-2ab+b^4$

30. x^2+xy+y^2

31. $a^3-2ab+b^2$

32. $x^3-2x^2y+xy^2$

If the monomial factor x is taken out, the other factor is of the type $a^2-2ab+b^2$, hence

$$x^3-2x^2y+xy^2\equiv x(x^2-2xy+y^2)\equiv x(x-y)^2.$$

In Exs. 33–40, find the prime factors by removing the monomial factor first.

33. $3x^2-6xy+3y^2$

34. $4ax^2+4a^2x+a^3$

35. $12x^2-36x+27$

36. x^3-2x^2+x

37. $a^3+ab^2-2a^2b$

38. $x^3y-4x^2y^2+4xy^3$

39. $4a^2+8ab+4b^2$

40. $9x^4+9y^4-18x^2y^2$

§ 57. Review of Types I–III.

EXERCISES

Find the prime factors of each of the following expressions, and check.

1. m^2-4n^2

2. $m^24m-n+4n^2$

3. m^2+4n^2+4mn

4. $9x^2-4y^2$

5. $9x^2-6xy+y^2$

6. a^2-1

7. a^2+2a+1

8. a^3-2a^2+a

9. $5x^2+5xy+5y^2$

10. b^2-6b+9

11. $25x^2-16$

12. $25x^2+40xy+16y^2$

13. y^2-y

14. y^2-2y+1

15. $2y^3-2y$

16. x^3-x^2+x

17. $(a-2b)^2-x^2$

18. $(2x-3)^2-16$

19. $(a^2-10)^2-36$

20. $(4p^2-1)^2-9$

21. $a^2-2ab+b^2-c^2$

SOLUTION.
$$\begin{aligned}a^2-2ab+b^2-c^2&\equiv(a^2-2ab+b^2)-c^2\\&\equiv(a-b)^2-c^2\\&\equiv(a-b+c)(a-b-c)\end{aligned}$$

22. $4a^2-4ab+b^2-25$

SOLUTION. $4a^2-4ab+b^2-25\equiv(2a-b)^2-25$

$\equiv(2a-b+5)(2a-b-5)$

23. $(a^2-6ab+9b^2)-x^2$

24. $(x^2-6x+9)-y^2$

25. $(b^2-10b+25)-16$

26. $m^2-4mn+4n^2-25$

27. $4a^2-12ab+9b^2-16c^2$

28. $(a^4-2a^2+1)-9$

§ 58. Trinomials — The Product of Two Binomials Having a Common Term.

TYPE IV. $\boldsymbol{x^2+(a+b)x+ab}$.

$$\boldsymbol{x^2+(a+b)x+ab\equiv(x+a)(x+b)}$$

EXAMPLE 1. Factor $x^2+8x+15$.

SOLUTION. Since $+15$ is the product of the two number terms, their signs are *like*.

Since $+8$ is the sum of the two number terms, their signs are *plus*, hence

$$x^2+8x+15\equiv(x+5)(x+3)$$

CHECK. Multiply $x+5$ by $x+3$.

EXAMPLE 2. Factor $x^2-8x+15$.

SOLUTION. Since $+15$ is the product of the two number terms, their signs are *like*.

Since -8 is the sum of the two number terms, their signs are *minus*, hence

$$x^2-8x+15\equiv(x-5)(x-3)$$

CHECK. Multiply $x-5$ by $x-3$.

EXAMPLE 3. Factor $x^2+2x-15$.

SOLUTION. Since -15 is the product of the two number terms, their signs are *unlike*.

Since $+2$ is the sum of the two number terms, the sign of the larger is *plus*, hence

$$x^2+2x-15\equiv(x+5)(x-3)$$

CHECK. Multiply $x+5$ by $x-3$.

EXAMPLE 4. Factor $x^2-2x-15$.

SOLUTION. Since -15 is the product of the two number terms, their signs are *unlike*.

Since -2 is the sum of the two number terms, the sign of the larger is *minus*, hence

$$x^2-2x-15 \equiv (x-5)(x+3)$$

CHECK. Multiply $x-5$ by $x+3$.

EXERCISES

Factor each of the following trinomials and check.

1. y^2+5y+6
2. $a^2-8a+15$
3. $a^2+2a-15$
4. m^2-m-6
5. m^2+m-6
6. $b^2+7b+12$
7. p^2+p-12
8. $x^2-9x+18$
9. c^2-c-20
10. $d^2+14d+33$
11. $h^2-14h-15$
12. k^2+6k+5
13. $m^2+11m-60$
14. $y^2-25y+150$
15. $b^2-3b-54$
16. $x^2y^2-4xy+3$
17. $a^2b^2-ab-132$
18. $c^2d^2-8cd-9$
19. y^2+2y+1
20. a^4-6a^2-16
21. x^2-x-3

This trinomial cannot be factored under Type IV. Why? In Exs. 22–31, some of the given trinomials cannot be factored under Type IV, while others can be. State why in each case.

22. a^2-a-30
23. m^2+3m-5
24. x^2+4x+5
25. p^2-3p+2
26. y^2-5y-6
27. b^2+5b+6
28. a^2+6b+8
29. x^2+x-2
30. x^2+x+1
31. x^3+3x+2

32. $3x^2-15x+18$

If the monomial factor 3 is taken out, the other factor is of Type IV.

$$3x^2-15x+18\equiv 3(x^2-5x+6)\equiv 3(x-3)(x-2)$$

In Exs. 33–40, find the *prime* factors.

33. $a^2b+6ab+8b$

34. $ay^2-14ay+24a$

35. $4t^2-8t-12$

36. x^3-6x^2+5x

37. $2a^3-10a^2+12a$

38. $a^2x^2-2a^2x-15a^2$

39. $y^4-13y^3+36y^2$

40. $2x^2+4x+2$

§ 59. Review of Types I–IV.

EXERCISES

Find the prime factors of each of the following expressions and check.

1. $6a^2+16a$

2. a^2-4y^2

3. $x^2+12x+36$

4. x^2-2x-8

5. $3x^2+12x+6$

6. n^2-4n+4

7. $4a^2+4a+1$

8. $4x^2-4xy+y^2$

9 x^3-x

10. $4n^2+4$

11. $c^2-4c-21$

12. $a^2b+7ab-18b$

13. a^4-1

14. $3x^2-3x-6$

15. a^4-a^2-6

16. $9a^2-18$

§ 60. Trinomials—The Product of Any Two Binomials.

Type V. $\boldsymbol{ax^2+bx+c}$.

$$\boldsymbol{ax^2+bx+c\equiv(?x+\,?)(?x+\,?)}$$

Example 1. Factor $4x^2-13x+10$.

Solution. Since $+10$ is the product of the two number terms, their signs are *like*.

Since $-13x$ is the sum of the cross-products, the sign of each number term is *minus*.

FACTORS OF $4x^2$	FACTORS OF $+10$
$\begin{cases} 4x \\ x \end{cases}$ or $\begin{cases} 2x \\ 2x \end{cases}$	$\begin{cases} -10 \\ -1 \end{cases}$ or $\begin{cases} -5 \\ -2 \end{cases}$

Since the sum of the cross-products must be $-13x$, the factors are:

$$\begin{array}{c} 4x-5 \\ \times \\ \underline{x-2} \\ -13x \end{array}$$

$$4x^2-13x+10 \equiv (4x-5)(x-2)$$

CHECK. Multiply $4x-5$ by $x-2$.

EXAMPLE 2. Factor $6y^2-5y-4$.

SOLUTION. Since -4 is the product of the two number terms, their signs are *unlike*.

Since $-5y$ is the sum of the cross-products, the sign of the greater cross-product is *minus*.

FACTORS OF $6y^2$	FACTORS OF -4
$\begin{cases} 6y \\ y \end{cases}$ or $\begin{cases} 3y \\ 2y \end{cases}$	$\begin{cases} -4 \\ +1 \end{cases}$ or $\begin{cases} +4 \\ -1 \end{cases}$ or $\begin{cases} -2 \\ +2 \end{cases}$ or $\begin{cases} +2 \\ -2 \end{cases}$

Since the sum of the cross-products must be $-5y$, the factors are;

$$\begin{array}{c} 3y-4 \\ \times \\ \underline{2y+1} \\ -5y \end{array}$$

$$6y^2-5y-4 \equiv (3y-4)(2y+1)$$

CHECK. Multiply $3y-4$ by $2y+1$.

EXERCISES

Factor each of the following trinomials and check.

1. $3x^2+5x+2$
2. $2x^2-5x+2$
3. $2y^2-y-28$
4. $6m^2+7m-3$
5. $6x^2-x-2$
6. $14a^2-39a+10$
7. $8x^2-10x+3$
8. $2r^2+r-15$
9. $7x^2-3x-4$
10. $3x^2+7x+2$
11. $5x^2-9x-2$
12. $3x^2+4x+1$
13. $8y^2+22y+15$
14. $8+22x+15x^2$

    Ans. $(2+3x)(4+5x)$.
15. $2-a-21a^2$
16. $15-r-2r^2$
17. $2-5x+2x^2$
18. $6m^2-19m+15$
19. $20a^2-a-99$
20. $6-5x-4x^2$
21. $6x^2+7x+2$

In Exs. 22–30, find the prime factors.

22. $4t^2+22t+10$
23. $20x-9x^2-20x^3$
24. $10r^2-5r-75$
25. $7x^3-x^2-6x$
26. $6x^2-33x+36$
27. $16x^2+16x-12$
28. $15x^2+21x+6$
29. $30x^2-35x+10$
30. $15x^2-21x+6$

§ 61. Summary of Factoring.

TYPE I. **Monomial Factors in Polynomials.**

$$am+bm-cm\equiv m(a+b-c)$$

TYPE II. **Binomials — The Difference of Two Squares.**

$$a^2-b^2\equiv(a+b)\ (a-b)$$

TYPE III. **Trinomials — The Squares of Binomials.**

$$a^2\pm 2ab+b^2\equiv(a\pm b)^2$$

TYPE IV. **Trinomials — The Product of Two Binomials Having a Common Term.**

$$x^2+(a+b)x+ab\equiv(x+a)(x+b)$$

TYPE V. **Trinomials — The Product of Any Two Binomials.**

$$ax^2+bx+c\equiv(?x+?)(?x+?)$$

MISCELLANEOUS EXERCISES IN FACTORING

Find the *prime* factors of each of the following expressions and check.

1. a^3+2a^2+2a
2. m^2-64y^2
3. p^2-4p+4
4. t^2-t-12
5. $6a^2+7a-3$
6. $3x^2+5x+2$
7. $3x^2+27x+42$
8. $a^4+36x^2+12a^2x$
9. $x^2-11x+24$
10. $m^2+11m-12$
11. $100-9x^2$
12. $x^2-17x+72$
13. y^4-15y^2-100
14. $5a^3+10a^3b^2+30a^3b^4$
15. x^4-2x^2-120
16. $a^2-a-240$
17. y^4-3y^2-180
18. $12a^2-23ab+10b^2$
19. $1-20y+75y^2$
20. $2n^3-n^2-3n$
21. $x^2+2xy-8y^2$
22. $1-16y^4$
23. $4x^2+4xy+y^2$
24. $x-x^2$
25. x^3+4x
26. $6t^2-7t-3$
27. $x^4+y^4-2x^2y^2$
28. $3a^2-12a-96$
29. $6x^2-15x+6$
30. $m^2-14m-95$
31. $1-4a+4a^2$
32. $1-2a+a^2$
33. $1-15xy+56x^2y^2$
34. $7-3x-4x^2$
35. a^3-4a
36. $2+2a^2$
37. a^8-y^8
38. $3x^4-23x^2-36$
39. $(2a-b)^2-c^2$
40. $(3y+z)^2-9$
41. $a^2-6a+9-y^2$
42. $4x^2-12xy+9y^2-9z^2$
43. $x^4-2x^2y^2+y^4-9z^4$
44. $(a+b)^2-3(a+b)+2$

§ 62. **Quadratic Equations.** In Chapter V the equations solved were *linear* equations. ***Linear equations*** are equations of the first degree; that is, they involve only the first power of the unknown number.

Quadratic equations are equations of the second degree; that is, they involve the *second*, but no higher, power of the unknown number.

EXERCISES

In the following exercises:

(*a*) State the degree of each equation.

(*b*) Select the linear and the quadratic equations.

1. $x-5=0$	6. $m^2-16=0$
2. $x^2+x=12$	7. $5x^2-9x=2$
3. $3a-4=0$	8. $p^2=12$
4. $a^2-5a=14$	9. $\frac{1}{2}t+5=0$
5. $6x^2-x=2$	10. $t^2-3t-10=0$

§ 63. **Solution of Quadratic Equations by Factoring.** To solve equations by factoring the following axiom (Axiom A) is necessary. *If the product of two or more factors is zero, at least one of the factors is zero.*

EXAMPLE 1. Solve the equation $x^2-5x=6$, and check the roots.

SOLUTION.

①	$x^2-5x=6$	
②	$x^2-5x-6=0$	① -6
③	$(x-6)(x+1)=0$	② $\equiv$ (Factoring)
④	$x-6=0$, or $x+1=0$	③ by Ax. A
⑤	$x=6$, or $x=-1$	

CHECK. $(6)^2-5(6)\stackrel{?}{=}6$ $\qquad$ $(-1)^2-5(-1)\stackrel{?}{=}6$

$36-30=6$ $\qquad$ $+1+5=6$

Ans. $x=6$, or -1.

EXAMPLE 2. Solve the equation $a^2+3a=10a+18$, and check the roots.

SOLUTION.

①	$a^2+3a=10a+18$	
②	$a^2-7a-18=0$	① $-10a-18$
③	$(a-9)(a+2)=0$	② $\equiv$ (Factoring)
④	$a-9=0$, or $a+2=0$	③ by Ax. A
⑤	$a=9$, or $a=-2$	

CHECK.

$$(9)^2+3(9)\overset{?}{=}10(9)+18 \qquad (-2)^2+3(-2)\overset{?}{=}10(-2)+18$$

$$81+27\overset{?}{=}90+18 \qquad +4-6\overset{?}{=}-20+18$$

$$108=108 \qquad -2=-2$$

Ans. $a=9$, or -2.

EXAMPLE 3. Solve the equation $4x+2=6x^2+3x$, and check the roots.

SOLUTION.

①	$4x+2=6x^2+3x$	
②	$0=6x^2-x-2$	① $-4x-2$
③	$0=(3x-2)(2x+1)$	② $\equiv$ (Factoring)
④	$3x-2=0$, or $2x+1=0$	③ by Ax. A
⑤	$3x=2$, or $2x=-1$	
⑥	$x=\frac{2}{3}$, or $x=-\frac{1}{2}$	

CHECK.

$$4(\tfrac{2}{3})+2\overset{?}{=}6(\tfrac{2}{3})^2+3(\tfrac{2}{3}) \qquad 4(-\tfrac{1}{2})+2\overset{?}{=}6(-\tfrac{1}{2})^2+3(-\tfrac{1}{2})$$

$$\tfrac{8}{3}+2\overset{?}{=}6(\tfrac{4}{9})+2 \qquad -2+2\overset{?}{=}6(\tfrac{1}{4})-\tfrac{3}{2}$$

$$4\tfrac{2}{3}=4\tfrac{2}{3} \qquad 0=0$$

Ans. $x=\frac{2}{3}$, or $-\frac{1}{2}$.

§ **64. Summary.** To solve a quadratic equation by factoring:

(1) *Collect all the terms in one member (leaving the other member zero).*

(2) *Factor that member.*

(3) *Write each factor equal to zero* (Axiom A).

(4) *Solve each equation thus formed.*

EXERCISES

Solve each of the following equations, and check the roots.

1. $x^2-7x+12=0$

2. $x^2+3x=10$

3. $y^2+8y=-7$

4. $2x^2-5x=-2$

5. $5x^2=4x+1$

6. $6x^2-11x=2$

7. $y^2=13y-36$

8. $x^2+100=20x$

9. $k(k-11)=-30$

10. $10p-3=3p^2$

11. $a(a+10)=-24-4a$

12. $2x^2-4x=-40+14x$

13. $m^2=10m-25$

14. $x^2-x=0$ *Ans.* $x=0$, or 1.

15. $a^2-3a=0$

16. $y^2=5y$

17. $c^2-4=0$

18. $4m^2-9=0$

19. $5a^2-10a=0$

20. $6t(t-1)=72$

21. $60x+4x^2+144=8x$

22. $3x^2-5x-2=0$

23. $18y=63-y^2$

24. $3m^2-6=-7m$

25. $a-2=-3a^2$

26. $\frac{x^2}{7}=x$

27. $\frac{t^2}{2}-t=4$

28. $p^2+3p=-\frac{5}{4}$

29. $3-\frac{7m}{4}=-\frac{m^2}{4}$

30. $y^2-\frac{11}{6}y=\frac{1}{3}$

PROBLEMS

In checking, substitute each result in the statement of the given problem.

1. Find two numbers whose difference is 9, and whose product is 90. (Let n and $n+9$ be the two numbers. Why?)

2. Find two numbers whose difference is 4, and whose product is 12.

3. Find two numbers whose sum is 14, and whose product is 33. (Let n and $14-n$ be the two numbers. Why?)

4. Find two numbers whose sum is 10, and whose product is 24.

5. The product of two consecutive integers is 20. Find them. (Let n and $n+1$ be the two consecutive integers. Why?)

6. The product of two consecutive integers is 90. Find them.

7. Find two consecutive integers the sum of whose squares is 25.

8. Find two even consecutive integers the sum of whose squares is 100.

9. Find a number whose square less 6 is equal to 5 times the number.

10. Find a number whose square increased by 8 is equal to 6 times the number.

11. The square of a number exceeds the number itself by 56. Find the number.

12. A rectangular room has an area of 240 sq. ft., one side being 8 ft. shorter than the other. Find the dimensions. Do both roots of the equation satisfy the problem?

13. A picture, 8″ by 12″, is placed in a frame of uniform width. If the area of the frame is the same as the area of the picture, what is the width of the frame? (Fig. 23.) Do both roots of the equation satisfy the problem?

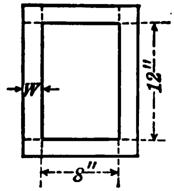

FIG. 23.

14. An open box is made from a square piece of tin by cutting out a 4-inch square from each corner and turning up the sides. How large is the original square if the box contains 64 cu. in.? (Fig. 24.) (Let x = width of piece of tin.)

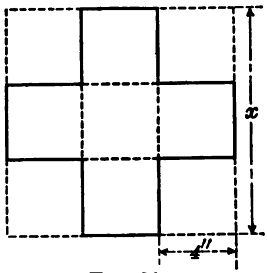

FIG. 24.

15. A fence 190 ft. long surrounds a rectangular field that contains 1800 sq. ft. Find the dimensions of the field.

16. A ventilator with a rectangular opening 20 in. high gives 64 sq. in. more space than one with a square opening of the same width. How wide is the ventilator?

17. A room is 15 feet square. On account of an error in measuring the dimensions of the floor, an area of 361 square inches too large was obtained. What was the error in the measurement (in inches)? [The equation is: $(n+180)^2=180^2+361$.]

18. A room is 3.2 meters square. On account of an error in measuring the dimensions of the floor, an area of 1276 square centimeters too small was obtained. What was the error in the measurement (in centimeters)?

CHAPTER XI

RADICALS AND ROOTS

§ 65. Rational and Irrational Numbers. For each of the quadratic equations in Chapter VI, the values of the unknown number were *rational.*

A ***rational number*** is an integer, or a number which is the quotient of two integers, for example: 8, -7, 0.875, $\frac{5}{8}$, $\sqrt{25}$.

For many quadratic equations the values of the unknown are *irrational;* for example, to find the side of a square whose area is 45 sq. in., the equation is $45 = s^2$, in which $s = \sqrt{45}$.

An ***irrational number*** is a number that is not rational; for example: $\sqrt{2}$, $\sqrt{12}$, $\sqrt[3]{5}$, $\sqrt[3]{1.7}$.

In these examples the symbol $\sqrt{}$ is called the *radical sign.* In the $\sqrt[3]{5}$, the 3 is the *index* of the root and shows that the cube root is required. When no index is written, the root required is the *square root.*

EXERCISE

Make a list of the rational numbers and of the irrational numbers from among those that follow. Name the index and root required for each irrational number. 5, -12, $\sqrt{3}$, $\frac{4}{9}$, 0.305, $\sqrt{10}$, $\sqrt[3]{4}$, $\frac{2}{3}$, $\sqrt{1.5}$, $\sqrt[3]{80}$, 0.04, $\sqrt{1.7}$, $\frac{5}{12}$, $\sqrt{16}$, $\sqrt[3]{27}$.

§ 66. Reduction of Radicals. When a radical is irrational, it is often convenient not to evaluate it, but to retain it in radical form. When this is to be done the radical should be reduced to its simplest form.

A radical, involving square root only, is in its simplest form when the number under the radical sign is integral, and when the number contains no factor that is a perfect square.

EXAMPLE 1. Reduce $\sqrt{50}$ to its simplest form.

SOLUTION. $\sqrt{50}=\sqrt{25\times 2}=5\sqrt{2}$. *Ans.* $5\sqrt{2}$.

EXAMPLE 2. Reduce $\sqrt{\frac{3}{8}}$ to its simplest form.

SOLUTION. $\sqrt{\frac{3}{8}}=\sqrt{\frac{6}{16}}=\sqrt{\frac{1}{16}\times 6}=\frac{1}{4}\sqrt{6}$. *Ans.* $\frac{1}{4}\sqrt{6}$.

EXAMPLE 3. Reduce $\sqrt{\dfrac{9a}{5bc^2}}$ to its simplest form.

SOLUTION.
$$\sqrt{\frac{9a}{5bc^2}}=\sqrt{\frac{45ab}{25b^2c^2}}=\sqrt{\frac{9}{25b^2c^2}\times 5ab}$$
$$=\frac{3}{5bc}\sqrt{5ab} \qquad \textit{Ans. } \frac{3}{5bc}\sqrt{5ab}.$$

Reduce each of the following radicals to its simplest form.

1. $\sqrt{12}$
2. $\sqrt{24}$
3. $\sqrt{49}$
4. $\sqrt{3a^2}$
5. $\sqrt{4a}$
6. $\sqrt{80}$
7. $\sqrt{200}$
8. $\sqrt{1000}$
9. $\sqrt{45a^2}$
10. $\sqrt{8x^2y}$
11. $\sqrt{2xy^2}$
12. $\sqrt{500a^2b^2}$
13. $\sqrt{\frac{2}{3}}$
14. $\sqrt{\frac{1}{2}}$
15. $\sqrt{\frac{1}{5}}$
16. $\sqrt{\dfrac{4a}{3b}}$
17. $\sqrt{\dfrac{5a^2}{2x}}$
18. $\sqrt{\dfrac{50x^2y}{27}}$

In Exs. 19–30, after reducing each radical, evaluate it. Use the following values: $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt{5}=2.236$, and $\sqrt{10}=3.162$.

19. $\sqrt{8}$
20. $\sqrt{18}$
21. $\sqrt{45}$
22. $\sqrt{300}$
23. $\sqrt{20}$
24. $\sqrt{75}$
25. $\sqrt{125}$
26. $\sqrt{1000}$
27. $\sqrt{\frac{1}{3}}$
28. $\sqrt{\frac{1}{10}}$
29. $\sqrt{\frac{2}{5}}$
30. $\sqrt{\frac{25}{2}}$

§ 67. Multiplication of Radicals. The square root of a number is one of the *two equal* factors of the number. Hence, when you square the square root of a number, the result must be the given number. For example, $(\sqrt{2})^2=2$, $(\sqrt{7})^2=7$, etc.

EXAMPLE 1. Find the product: $2\sqrt{3}\times 5\sqrt{3}$.

SOLUTION. $2\sqrt{3}\times 5\sqrt{3}=10(\sqrt{3})^2=10\times 3=30$.

Ans. 30.

EXAMPLE 2. Find the product: $\sqrt{3}(2\sqrt{3}-5)$.

SOLUTION.

$$\sqrt{3}(2\sqrt{3}-5)=2(\sqrt{3})^2-5\sqrt{3}=2\times 3-5\sqrt{3}=6-5\sqrt{3}.$$

Ans. $6-5\sqrt{3}$.

EXERCISES

In each of the following exercises, find the product.

1. $\sqrt{5}\times\sqrt{5}$
2. $3\sqrt{2}\times\sqrt{2}$
3. $4\sqrt{10}\times 3\sqrt{10}$
4. $\sqrt{18}\times\sqrt{8}$
5. $\sqrt{2}(3-\sqrt{2})$
6. $\sqrt{5}(2\sqrt{5}-1)$
7. $2\sqrt{3}(\sqrt{3}-3)$
8. $(2+\sqrt{3})(2-\sqrt{3})$
9. $(2+\sqrt{3})(2+\sqrt{3})$
10. $(\sqrt{5}-2)(\sqrt{5}-3)$

PROBLEMS

In each of the following problems, leave the result in radical form; reduce the radical to its simplest form unless required to evaluate it.

1. The sides of a right triangle are 4″ and 2″. Find the hypotenuse. (See Ex. 7, page 26.)
2. The side of a square is 5″. Find its diagonal.
3. The side of a square is 4″. Find its diagonal.
4. The side of a square is 10″. Find its diagonal.
5. The side of a square is a inches. Find its diagonal.

6. The side of an equilateral triangle is 10″. (*a*) Find its altitude. (See Ex. 14, page 41.) (*b*) Find its area.

7. The side of an equilateral triangle is 4″. (*a*) Find its altitude. (*b*) Find its area.

8. The side of an equilateral triangle is 12″. (*a*) Find its altitude. (*b*) Find its area.

9. The side of an equilateral triangle is a inches. (*a*) Find its altitude. (*b*) Find its area.

10. The sides of a right triangle are $\sqrt{3}$ in. and $\sqrt{2}$ in. Find its hypotenuse (to three figures).

11. The sides of a rectangle are $\sqrt{7}$ in. and $\sqrt{3}$ in. Find its diagonal (to three figures).

12. The side of a square is $2\sqrt{2}$ in. Find its diagonal.

§ 68. Table of Squares and Square Roots. In Chapter II you found the square root of a number by estimating one of its two equal factors, and then by dividing the given number by your estimate.

It is customary for those who need to evaluate square roots, cube roots, etc., frequently, to consult tables.

On pages 288–290 you will find the squares of all numbers of three figures from 1.00 to 10.00. This table may also be used for finding the square roots of numbers. When using the table, make a mental estimate of your result in order to check the location of the decimal point.

A study of the examples on pages 285–287 will make it clear how to use the table for finding the square and the square root of any number of three figures.

EXERCISES

Copy the following table and find the squares and square roots to three figures using the table on pages 288–290. Make the estimates for all the exercises first.

Number	Est. Square	Square	Est. Square Root	Square Root
1. 5.43				
2. 54.3				
3. 0.543				
4. 1.27				
5. 12.7				
6. 127.				
7. 0.127				
8. 6.40				
9. 64.0				
10. 640.				
11. 0.640				
12. 0.0640				
13. 3				
14. 30				
15. 300				

PROBLEMS

Check the answer to each of the following problems in the statement of the given problem. Record the answer to *three figures*, when it is not exact.

1. The area of a square room is 192 sq. ft. What is its length?

2. The sides of a right triangle are 8.00″ and 11.0″. Find its hypotenuse. (Formula, Ex. 7, page 26.)

3. The sides of a right triangle are 22.4 cm. and 18.6 cm. Find its hypotenuse.

4. The sides of a square are each 12.0″. Find the diagonal.

5. A trunk is 41.3″ by 23.8″ (inside). What is the longest cane that can be placed in the bottom of it?

6. The hypotenuse of a right triangle is 20″ and one side is 12″. Find the other side.

SOLUTION. The formula is:

①	$a^2 = c^2 - b^2$	
②	$a^2 = (c+b)(c-b)$	①≡ (Factoring)
③	$a^2 = 32 \times 8$	②≡ (Substituting)
④	$a^2 = 256$	③≡
⑤	$a = 16$	④√

Ans. 16″.

7. The hypotenuse of a right triangle is 12.3″ and one side is 11.8″. Find the other side.

8. A ladder is 38′ long and just reaches a window. If its foot is 13′ from the building, how high is the window?

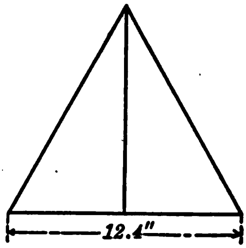

FIG. 25.

9. Find the altitude of an equilateral triangle whose side is 12.4″ (Fig. 25).

10. Find the altitude of an equilateral triangle whose side is 18.6 cm. Find its area.

11. Each side of a regular hexagon is 8.6″ long (Fig. 26). Find its area.

(SUGGESTION. Find the area of one triangle first. The triangles are equilateral. Why?)

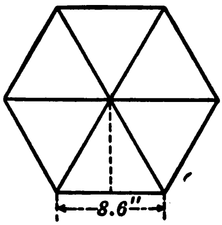

FIG. 26.

12. Each side of a regular hexagon is 27.6 cm. Find its area.

13. A metal plate 10.3″ square is required to be cut from a circular plate. What is the diameter of the smallest circular plate that can be used?

14. What is the diameter of a circle whose area is 47.4 sq. in.?

15. The inside cross-sectional area of a water pipe is 1.52 sq. cm. What is its diameter?

(In Exs. 16–25, change the subject of the formula as required.)

16. $A=\pi r^2$, $r=?$

17. $s=\frac{1}{2}gt^2$, $t=?$

18. $V=\pi r^2h$, $r=?$

19. $V=\frac{1}{3}\pi r^2h$, $r=?$

20. $S=4\pi r^2$, $r=?$

21. $F=\frac{mv^2}{2g}$, $v=?$

22. $c^2=a^2+b^2$, $a=?$

23. $H=\frac{d^2n}{2.5}$, $d=?$

24. $R=\frac{kl}{d^2}$, $d=?$

25. $A=\frac{s^2}{4}\sqrt{3}$, $s=?$

CHAPTER XII

QUADRATIC EQUATIONS

§ 69. Types of Quadratic Equations. The equation $ax^2+bx+c=0$ is called a *complete quadratic equation*, because it contains one term involving x^2, one term involving x, and a third term free from x.

Quadratic equations of the forms

$$x^2+c=0$$
$$x^2+bx=0$$

are called *incomplete quadratic equations*, because either the term involving x, or the term free from x is missing.

EXERCISES

From the following equations make a list of the complete and of the incomplete quadratic equations.

1. $5x^2+3x-7=0$
2. $x^2-7=0$
3. $x^2-5x-2=0$
4. $5a^2-a=0$
5. $7y^2-15=0$
6. $3b^2+5b+3=0$
7. $12a^2=0$
8. $m^2-m=0$

§ 70. Solution of Incomplete Quadratic Equations.

EXAMPLE 1. Solve $x^2=16$, and check the roots.

SOLUTION A (by factoring).

①	$x^2=16$	
②	$x^2-16=0$	①-16
③	$(x-4)(x+4)=0$	②$\equiv$ (Factoring)
④	$x-4=0$, or $x+4=0$	③ by Ax. A
⑤	$x=4$, or $x=-4$	

From this solution you should note that the equation $x^2-16=0$ has *two* roots, namely $+4$ and -4.

SOLUTION B (by square root).

① $x^2=16$

② $x=+4$, or -4 ①$\surd$

Solution A shows that it is necessary to write both $+4$ and -4 for the square root of 16.

CHECK. $(+4)^2=+16$ $(-4)^2=+16$

Ans. $x=+4$, or -4.

It follows that the square root of a number may be either *plus* or *minus*. The *plus* square root is called the *principal root.*

EXAMPLE 2. Solve $x^2-7=0$, and check the roots.

SOLUTION A (by factoring).

① $x^2-7=0$

Since 7 is not a perfect square, its two equal factors are *irrational.*

② $(x-\sqrt{7})(x+\sqrt{7})=0$ ①$\equiv$(Factoring)

③ $x-\sqrt{7}=0$, or $x+\sqrt{7}=0$ ② by Ax. A

④ $x=+\sqrt{7}$, or $x=-\sqrt{7}$

⑤ $x=+2.65$, or $x=-2.65$

SOLUTION B (by square root).

① $x^2-7=0$

② $x^2=7$ ①$+7$

③ $x=+2.65$, or -2.65 ②$\equiv$(Table)

CHECK. (Use Table of Squares.)

$(+2.65)^2-7\stackrel{?}{=}0$ $(-2.65)^2-7\stackrel{?}{=}0$

$7.02-7\stackrel{?}{=}0$ $7.02-7\stackrel{?}{=}0$

Ans. $x=+2.65$, or -2.65.

The discussion of these two methods and the interpretation of the checks follows:

Solution A shows that both the *plus* and the *minus* values of $\sqrt{7}$ must be recorded. The check shows that *both* values satisfy the equation.

Method B requires less labor, but it is essential that you record *both* signs in your answer. The answer may be written: $x = \pm 2.65$.

Note. In the check the two members are not identical, because the square root of 7 is only an approximate value. The more figures used in the square root, the more nearly identical would the members of the check become. When irrational roots are evaluated, the two members in the "Check" will not be identical.

Example 3. Solve the equation $\frac{3x^2}{4} = 11$, and check the roots.

Solution.

①	$\frac{3x^2}{4} = 11$	
②	$3x^2 = 44$	① × 4
③	$x^2 = 14.7$	② ÷ 3
④	$x = \pm 3.83$	③ $\sqrt{\ }$

Since the tables give the square roots of three-figure numbers only, retain not more than three figures in ③ and ④.

Check. (Use Table of Squares.)

$$\frac{3(3.83)^2}{4} \overset{?}{=} 11 \qquad \frac{3(-3.83)^2}{4} \overset{?}{=} 11$$

$$\frac{3 \times 14.7}{4} \overset{?}{=} 11 \qquad \frac{3 \times 14.7}{4} \overset{?}{=} 11$$

$$11.0 \overset{?}{=} 11 \qquad 11.0 \overset{?}{=} 11$$

Ans. $x = \pm 3.83$.

Note. The members of this check appear to be identical, but if the fourth figure were written, they would differ, that is, $\frac{3 \times 14.7}{4} = 11.02$.

EXAMPLE 4. Solve the proportion $\frac{12}{m}=\frac{m}{32}$, and check the roots.

SOLUTION.

①	$\frac{12}{m}=\frac{m}{32}$	
②	$32\cancel{m}\left(\frac{12}{\cancel{m}}\right)=\cancel{32}m\left(\frac{m}{\cancel{32}}\right)$	① $\times 32m$
③	$384=m^2$	② $\equiv$
④	$\pm 19.6=m$	③ $\sqrt{\ }$

CHECK. $\frac{12}{19.6}\overset{?}{=}\frac{19.6}{32}$ $\qquad$ $\frac{12}{-19.6}\overset{?}{=}\frac{-19.6}{32}$

$0.612\overset{?}{=}0.612$ $\qquad$ $-0.612\overset{?}{=}-0.612$

Ans. $m=\pm 19.6$.

NOTE. The equation $\frac{12}{m}=\frac{m}{32}$ is a *proportion*, since it expresses the equality of two ratios. In this proportion, m is called the *mean* proportional between 12 and 32.

EXERCISES

Find, to three figures, *two* roots for each of the following equations, and check each root.

1. $3a^2=96.6$
2. $9=4.5m^2$
3. $16=\frac{2x^2}{5}$
4. $\frac{3y^2}{4}=17.5$
5. $\frac{3}{5}=\frac{2a^2}{4}$
6. $5k^2+4=18$
7. $10(a^2+1)=18$
8. $2(3b^2-1)=4.66$
9. $\frac{2a^2+1}{3}=0.8$
10. $48=3.14r^2$
11. $\frac{4}{p}=\frac{p}{20}$
12. $\frac{5}{d}=\frac{d}{8}$
13. $\frac{10}{y}=\frac{y}{3}$
14. $\frac{16}{a}=\frac{a}{6}$
15. $\frac{3.5}{d}=\frac{d}{12}$
16. $\frac{2.4}{x}=\frac{x}{10.5}$

17. $\frac{17.2}{m}=\frac{m}{18}$

18. $\frac{37.3}{k}=\frac{k}{10}$

19. $\frac{41.6}{a}=\frac{a}{4}$

20. $\frac{1.12}{b}=\frac{b}{0.61}$

PROBLEMS

Check the answers to each of these problems in the statement of the given problem.

1. How long will it take a bomb to fall from an airplane 3000 feet high? (Use the formula $s=16t^2$.) Will both values of t satisfy the conditions?

2. How long will it take a bomb to fall from an airplane 2000 meters high?

3. The horse-power of a 4-cylinder automobile engine is 35. What is the diameter of its cylinders in inches? (Formula, page 37, Ex. 5.)

4. Write the formula for the statement: *The areas of two similar surfaces have the same ratio as the squares of any two corresponding lines.*

The formula is:

$$\frac{A_1}{A_2}=\frac{l_1^2}{l_2^2}$$

Explain the meaning of each letter. (Use the formula in Prob. 4 for Probs. 5–7.)

5. The areas of two circles are 40 sq. in. and 60 sq. in. The diameter of the smaller is 7.14 in. Find the diameter of the larger.

6. The areas of two similar triangles are 50 sq. in. and 30 sq. in. The altitude of the second is 15 in. Find the altitude of the other.

7. The surfaces of two balls are 16 sq. cm. and 40 sq. cm. What is the diameter of the smaller one, if the diameter of the larger is 3.57 cm.?

§ 71. Completing the Square. In order to solve quadratic equations of the type $ax^2+bx+c=0$, that have irrational roots, you must consider again how to complete a trinomial square. (See page 89.)

From $(x+a)^2 \equiv x^2+2ax+a^2$ you note that the third term of the trinomial is the square of half the coefficient of x in the second term. Hence, if you are given only the two terms x^2+2ax, you have to add a^2 in order to *complete* the square.

Figure 27 is a geometric illustration of the incomplete square x^2+2ax.

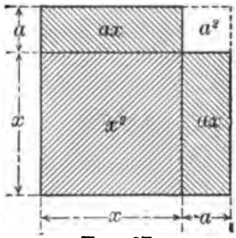

FIG. 27.

EXAMPLE 1. Complete the square in x^2+10x.

Half the coefficient of $x=5$.

$$5^2=25.$$

The complete square is $x^2+10x+25$.

EXAMPLE 2. Complete the square in $x^2-\frac{5}{3}x$.

Half the coefficient of $x = -\frac{5}{6}$.

$$(-\tfrac{5}{6})^2=\tfrac{25}{36}.$$

The complete square is $x^2-\frac{5}{3}x+\frac{25}{36}$.

EXERCISES

Complete the square in each of the following.

1. x^2+6x	**7.** x^2-5x
2. x^2+12x	**8.** x^2+x
3. x^2-8x	**9.** x^2-x
4. x^2-10x	**10.** x^2+11x
5. x^2+3x	**11.** $x^2+\frac{2}{3}x$
6. x^2+7x	**12.** $x^2+\frac{4}{5}x$

13. $x^2-\frac{2}{5}x$

14. $x^2-\frac{10}{3}x$

15. $x^2+\frac{3}{4}x$

16. $x^2-\frac{3}{5}x$

17. $x^2+\frac{1}{2}x$

18. $x^2-\frac{1}{3}x$

19. x^2+mx

20. x^2-bx

§ 72. Solution of Complete Quadratic Equations.

EXAMPLE 1. Solve the equation $x^2+6x=3$, and check the roots.

SOLUTION A (by completing the square and taking the square root of each member).

①	$x^2+6x=3$	
②	$x^2+6x+9=12$	①$+3^2$
③	$x+3=\pm 3.46$	②$\surd$ (Table)
④	$x=+3.46-3$, or $-3.46-3$	③-3
⑤	$x=0.46$, or -6.46	④$\equiv$

SOLUTION B (by completing the square and factoring).

①	$x^2+6x=3$	
②	$x^2+6x-3=0$	①-3
③	$(x^2+6x+9)-12=0$	②$\equiv$(Completing the square)
④	$(x+3-\sqrt{12})(x+3+\sqrt{12})=0$	③$\equiv$(Factoring)
⑤	$x+3-\sqrt{12}=0$, or	
	$x+3+\sqrt{12}=0$	④ by Ax. A
⑥	$x-0.46=0$, or	
	$x+6.46=0$	⑤$\equiv$(Table, collecting terms)
⑦	$x=+0.46$, or $x=-6.46$	

Solution B shows the necessity of using both + and − for $\sqrt{12}$ in equation ③, Solution A.

CHECK.

$$(0.46)^2+6(0.46)\overset{?}{=}3 \qquad (-6.46)^2+6(-6.46)\overset{?}{=}3$$
$$.212+2.76\overset{?}{=}3 \qquad 41.7-38.8\overset{?}{=}3$$
$$2.97\overset{?}{=}3 \qquad 2.9\overset{?}{=}3$$

Ans. $x=0.46$, or -6.46.

Why are the two numbers in these checks not identical?

EXAMPLE 2. Solve the equation $2x^2-3x=7$, and check the roots.

SOLUTION. Before completing the square, divide by the coefficient of x^2.

①	$2x^2-3x=7$	
②	$x^2-\frac{3}{2}x=\frac{7}{2}$	① $\div 2$
③	$x^2-\frac{3}{2}x+\frac{9}{16}=\frac{7}{2}+\frac{9}{16}$	② $+(\frac{3}{4})^2$
④	$x^2-\frac{3}{2}x+\frac{9}{16}=\frac{65}{16}$	③ $\equiv$
⑤	$x-\dfrac{3}{4}=\pm\dfrac{8.06}{4}$	④ $\sqrt{\ }$
⑥	$x=\dfrac{3}{4}+\dfrac{8.06}{4}$, or $\dfrac{3}{4}-\dfrac{8.06}{4}$	⑤ $+\dfrac{3}{4}$
⑦	$x=2.76$, or -1.26	⑥ $\equiv$

CHECK.

$$2(2.76)^2-3(2.76)\overset{?}{=}7 \qquad 2(-1.26)^2-3(-1.26)\overset{?}{=}7$$
$$2(7.62)-8.28\overset{?}{=}7 \qquad 2(1.59)+3.78\overset{?}{=}7$$
$$15.24-8.28\overset{?}{=}7 \qquad 3.18+3.78\overset{?}{=}7$$
$$6.96\overset{?}{=}7 \qquad 6.96\overset{?}{=}7$$

Ans. $x=2.76$, or -1.26.

§ **73. Summary.** To solve a quadratic equation by completing the square:

(1) *Write the equation in the form* $x^2+bx=c$.

(2) *Complete the square by adding the square of half the coefficient of* x *to each member.*

(3) *Take the square root of each member of this equation; write both the plus and the minus sign in front of the member containing the number term only.*

(4) *Solve each of the first degree equations thus formed.*

EXERCISES

Find *two* roots for each of the following equations by completing the square, and check each root. (For the equations in Exs. 1–16 the roots are rational.)

1. $x^2+2x=8$

2. $y^2+4y=12$

3. $x^2-2x=15$

4. $m^2-6m=-8$

5. $b^2+3b=10$

6. $a^2+a-12=0$

7. $x^2+\frac{7}{3}x-2=0$

8. $y^2-\frac{5}{2}y=\frac{3}{2}$

9. $2x^2+3x=2$

10. $3a^2+5a=-2$

11. $2y^2-y=28$

12. $3b^2+7b+2=0$

13. $4x^2-4x=-1$

14. $6p^2=p+2$

15. $2x^2=5x-2$

16. $3m^2+1=-4m.$

(For the equations in Exs. 17–30 the roots are irrational. Find each root to three figures.)

17. $x^2+4x=8$

18. $m^2+6m=3$

19. $y^2-2y=4$

20. $x^2-8x=-14$

21. $p^2=2+4p$

22. $5-6a=a^2$

23. $1-x=x^2$

24. $3x^2+4x=1$

25. $5y^2-2y=2$

26. $2m^2+3m=1$

27. $2a^2=5-5a$

28. $4d^2-8d+1=0$

29. $R(R+1)=3$

30. $5=2a(a+3)$

31. Solve $ax^2+bx+c=0$ for x by completing the square.

§ 74. Solution of Quadratic Equations by the Formula. The complete quadratic equation is $ax^2+bx+c=0$.

The solution of the equation $ax^2+bx+c=0$ by completing the square is as follows:

①	$ax^2+bx+c=0$	
②	$ax^2+bx=-c$	① $-c$
③	$x^2+\frac{b}{a}x=-\frac{c}{a}$	② $\div a$
④	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$	③ $+\left(\frac{b}{2a}\right)^2$
⑤	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$	④ ≡ (Combining fractions in second member)
⑥	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$	⑤ $\sqrt{\ }$
⑦	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$	⑥ $-\frac{b}{2a}$
⑧	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	⑦ ≡ (Combining fractions)

Hence the solutions of the equation

$$ax^2+bx+c=0,$$

are

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a}, \text{ or } \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Ans. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}.$

EXAMPLE 1. Solve $2x^2+7x+6=0$ by the formula, and check the roots.

SOLUTION. ① $2x^2+7x+6=0$

In equation ①, $a=+2$, $b=+7$, and $c=+6$.

Substituting these values for a, b, and c in the formula,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

② $$x=\frac{-7\pm\sqrt{49-48}}{4}$$

③ $$x=\frac{-7+1}{4},\text{ or }\frac{-7-1}{4} \qquad ②\equiv$$

④ $$x=-1\tfrac{1}{2},\text{ or }-2 \qquad ③\equiv$$

CHECK.

$$2(-\tfrac{3}{2})^2+7(-\tfrac{3}{2})+6\stackrel{?}{=}0 \qquad 2(-2)^2+7(-2)+6\stackrel{?}{=}0$$

$$\tfrac{9}{2}-\tfrac{21}{2}+6\stackrel{?}{=}0 \qquad 8-14+6=0$$

$$0=0$$

Ans. $x=-1\frac{1}{2}$, or -2.

EXAMPLE 2. Solve $2x^2+3x=2$ by the formula, and check the roots.

SOLUTION. ① $2x^2+3x=2$

② $2x^2+3x-2=0$ ①-2

In equation ②, $a=+2$, $b=+3$, and $c=-2$.

Substituting these values for a, b, and c in the formula,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

③ $$x=\frac{-3\pm\sqrt{9+16}}{4}$$

④ $$x=\frac{-3+5}{4},\text{ or }\frac{-3-5}{4} \qquad ③\equiv$$

⑤ $$x=\tfrac{1}{2},\text{ or }-2 \qquad ④\equiv$$

CHECK. $2(\frac{1}{2})^2+3(\frac{1}{2})\overset{?}{=}2$ $\quad$ $2(-2)^2+3(-2)\overset{?}{=}2$

$\frac{1}{2}+\frac{3}{2}=2$ $\quad$ $8-6=2$

Ans. $x=\frac{1}{2}$, or -2.

EXAMPLE 3. Solve $5x^2=2+2x$ by the formula, and check the roots.

SOLUTION. ① $\quad 5x^2=2+2x$

② $\quad 5x^2-2x-2=0 \quad$ ①$-2-2x$

In equation ②, $a=+5$, $b=-2$, and $c=-2$.

Substituting these values for a, b, and c in the formula,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

③ $$x=\frac{2\pm\sqrt{4+40}}{10}$$

④ $$x=\frac{2+6.63}{10}, \text{ or } \frac{2-6.63}{10} \qquad ③\equiv$$

⑤ $$x=0.863, \text{ or } -0.463 \qquad ④\equiv$$

CHECK.

$5(.863)^2\overset{?}{=}2+2(.863)$ $\quad$ $5(-.463)^2\overset{?}{=}2+2(-.463)$

$5(.745)\overset{?}{=}2+1.726$ $\quad$ $5(.214)\overset{?}{=}2-.926$

$3.725\overset{?}{=}3.726$ $\quad$ $1.070\overset{?}{=}1.074$

Ans. $x=0.863$, or -0.463.

§ **75. Summary.** To solve a quadratic equation by the formula:

(1) *Write it in the form* $ax^2+bx+c=0$.

(2) *Substitute the values for* a, b, *and* c *in the formula,*

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

(3) *Simplify the values of* x *thus obtained.*

EXERCISES

A. Find *two* roots for each of the following equations, using the formula. Check each root.

1. $x^2+4x+3=0$
2. $2x^2+x-28=0$
3. $x^2-6x=-8$
4. $6x^2-x=2$
5. $2x^2-11x+2=0$
6. $x^2+6x=-9$
7. $y^2-8y+14=0$
8. $a^2+6a=5$
9. $3y^2-4y=1$
10. $2m^2+5m=5$
11. $3x^2-5=5x-x^2+2$
12. $6x^2-7x-5=3x+5$
13. $\frac{m}{2}+\frac{1}{2}=\frac{1}{m}$
14. $15R+4=\frac{3}{R}$.
15. $1+\frac{22}{A}=6A$
16. $5a-8=\frac{21}{a}$
17. $\frac{3y-10}{14}=\frac{y+120}{y}$
18. $n-\frac{1}{n}=1\frac{1}{2}$
19. $m+3=\frac{4}{m-1}$
20. $x+2=\frac{3x^2-4}{2x+1}$

B. Find *two* roots for each of the following equations: (*a*) By factoring. (*b*) By completing the square. (*c*) By the formula.

1. $a^2-7a+12=0$
2. $y^2+3y=10$
3. $b^2+9b+14=0$
4. $m^2-6m=-9$
5. $3x^2+10x+3=0$
6. $5y^2-6y=11$
7. $2x^2+3x+1=0$
8. $9a^2-6a=-1$

PROBLEMS

1. Find two numbers whose difference is 6 and whose product is 135.

2. Find two numbers whose sum is 24 and whose product is 95.

3. Find two consecutive integers whose product is 72.

4. One side of a right triangle is 7″ longer than the other side. The hypotenuse is 13″. Find the two sides.

5. One side of a right triangle is 2 cm. longer than the other side. The hypotenuse is $\sqrt{34}$ cm. Find the two sides.

6. One side of a right triangle is 5″ less than the hypotenuse. The other side is $5\sqrt{3}''$. Find the hypotenuse.

7. A rectangle is 18″ by 24″. If the diagonal is to be increased 4″ in length, what will be the length of the rectangle, the width remaining the same?

8. The area of a square in square feet and its perimeter in inches are expressed by the same number. Find one side.

9. The area of a square in square centimeters and its perimeter in centimeters are expressed by the same number. Find one side.

10. Around a rectangular flower bed, 12′ by 16′, there is a border of turf which is everywhere of equal width, and whose area is 4 times the area of the bed. How wide is the turf to the nearest tenth of a foot?

11. $n=\dfrac{s(s-1)}{2}$ is the formula for finding the total number of different connections possible in a telephone exchange, given the number of subscribers s. If the total number of connections in a certain exchange is 4851, find the number of subscribers.

12. A photograph 8″ by 10″ is enlarged to twice its original area. What are the dimensions of the enlarged photograph to the nearest hundredth of an inch?

13. A photograph 6 cm. by 8 cm. is enlarged to $2\frac{1}{2}$ times its original area. What are the dimensions of the enlarged photograph to the nearest millimeter?

CHAPTER XIII

RATIO, PROPORTION, AND VARIATION

§ 76. **Ratio.** A *ratio* is the expression of the quotient of two quantities. The quantities must be of the same kind.

The ratio of a 12″ line to a 6″ line is $\frac{12}{6}$, or 2 (an *integer*). The ratio of a 5-lb. weight to a 12-lb. weight is $\frac{5}{12}$ (a *common fraction*). The ratio of the weight of a cubic foot of ice (57.5 lb.) to the weight of a cubic foot of water (62.5 lb.) is $\frac{57.5}{62.5}$, or .92 (a *decimal fraction*). The ratio of the diagonal of a square to one side is $\sqrt{2}$ (an irrational number).

EXERCISES

1. Draw two lines of different lengths.

(*a*) Measure them to the nearest hundredth of an inch and find the ratio of the first to the second.

(*b*) Measure them with a metric rule to the nearest millimeter, and find the ratio of the first to the second.

(*c*) Test the two ratios obtained in (*a*) and (*b*) for equality.

2. Two weights are $12\frac{1}{2}$ lb. and $7\frac{1}{2}$ lb.

(*a*) Find the ratio of the second to the first.

(*b*) Express the weights in ounces, and find the ratio of the second to the first.

(*c*) Test the two ratios obtained in (*a*) and (*b*) for equality.

3. Two pieces of iron weigh 7 lb. and 10 lb.

(*a*) Find the ratio of the first to the second.

(*b*) Express the weights in kilograms. (See page 46.) Find the ratio of the first to the second.

(*c*) Test the two ratios for equality.

NOTE. The ratio of two quantities remains the same whatever unit of measure is used.

4. The areas of two rectangles are 45 sq. in. and 12 sq. in. What is the ratio of the first to the second? Of the second to the first?

5. The volumes of two balls are 250 cu. cm. and 450 cu. cm. What is the ratio of the first to the second?

6. Two balls weigh $16\frac{1}{2}$ oz. and $7\frac{1}{4}$ oz. What is the ratio of the first to the second?

7. One line is 1.4 meters and another is 40 centimeters. What is the ratio of the first to the second?

8. Two weights are $4\frac{1}{2}$ lb. and 21 oz. Find the ratio of the second to the first.

9. The area of a lot of land is 4000 sq. ft. The area of the plan of this lot is 4 sq. in. What is the ratio of the area of the lot to the area of the plan?

10. A square room is 40 ft. on each side. The plan of the room, drawn to scale, is 2 in. on each side.

(*a*) Find the area of the room.

(*b*) Find the area of the plan.

(*c*) What is the ratio of the area of the room to the area of the plan?

§ **77. Proportion.** A ***proportion*** expresses the equality of two ratios. Hence a proportion is an equation each of whose members is a ratio. Usually the ratios in a proportion are common fractions. Exercises 35–60, pages 53 and 54, and Exercises 11–20, page 141, are proportions.

The first and last terms of a proportion are called the ***extremes;*** the second and third terms are called the ***means***.

For example, in the proportion $\frac{2}{5}=\frac{10}{25}$, 2 and 25 are the *extremes*, and 5 and 10 are the *means*.

In the proportion $\dfrac{a}{b}=\dfrac{b}{c}$, these means are the same number. Then that number, b, is called the ***mean proportional*** between a and c.

EXERCISES

Select from Exercises 1–10 those statements that are true proportions.

1. $\frac{3}{4}\stackrel{?}{=}\frac{9}{12}$

2. $\frac{2}{5}\stackrel{?}{=}\frac{6}{15}$

3. $\frac{3}{5}\stackrel{?}{=}\frac{7}{12}$

4. $\frac{5}{8}\stackrel{?}{=}\frac{7}{10}$

5. $\frac{1}{3}\stackrel{?}{=}\frac{3}{9}$

6. $\frac{4}{5}\stackrel{?}{=}\frac{5}{6}$

7. $\frac{4}{6}\stackrel{?}{=}\frac{6}{9}$

8. $\dfrac{2}{5}\stackrel{?}{=}\dfrac{5}{12.5}$

9. $\dfrac{3}{2.4}\stackrel{?}{=}\dfrac{5}{4}$

10. $\dfrac{3.6}{8}\stackrel{?}{=}\dfrac{1.8}{4}$

11. Select from Exercises 1–10 the proportions that contain a mean proportional and name the mean proportional in each case.

12. Solve the proportions 11–20, page 141.

13. Separate 84 into two parts which are in the ratio 3 to 11. (Let $n=$ the larger part; form an equation of the two ratios.)

14. Separate 240 into two parts which are in the ratio of 7 to 5.

15. Find the mean proportional between 18.4 and 30.5.

16. Brass is an alloy consisting of two parts copper and one part zinc. How many ounces of copper and zinc are there in 2 pounds of brass?

17. Gun metal consists of nine parts copper and one part zinc. How many ounces of each are there in 22 ounces of gun metal?

18. The total area of land on the earth is to the total area of water as 7 is to 18. If the total surface of the earth is 197,000,000 square miles, find the number of square miles of land and of water (to three figures).

19. A study of family budgets shows that on a salary of $900 per year, $405 is required for food. At the same rate how much would be required for food when the salary is $1000 per year?

20. On a salary of $850 per year, $168 is required for rent. At the same rate how much would be required for rent when the salary is $950 per year (to the nearest dollar)?

21. When one pound of substitutes must be purchased with every four pounds of wheat flour, how many pounds of substitutes must be purchased with ten pounds of wheat flour?

22. When two pounds of substitutes must be purchased with every three pounds of rye flour, how many pounds of substitutes must be purchased with ten pounds of rye flour (to the nearest pound)?

23. A 1-lb. loaf of bread contains $\frac{3}{4}$ as much nutrition as 1 pound of rice. When a 1-lb. loaf of bread costs 10 cents and 1 pound of rice costs 12 cents, which is the cheaper?

§ 78. Variables in Arithmetic.

Example 1. When sugar is selling at 9 cents a pound, 2 pounds will cost 18 cents, 3 pounds will cost 27 cents, 4 pounds will cost 36 cents, etc. The total cost *depends upon* the number of pounds purchased. As the number of pounds (n) increases, the cost (c) increases in the same ratio. In this illustration the total cost and the number of pounds are *variables*, and the price per pound (p) is a *constant*. The total cost is said to ***vary directly as*** the number of pounds. The relation may be expressed by the formula

$$c=np.$$

Example 2. A sum of $200 is loaned, the annual rate of interest being 6%. The interest for 1 month is $1, for 2 months it is $2, etc. The interest that accumulates *depends upon* the number of months. As the time (t) increases, the interest (i) increases in the same ratio. In this illustration the interest and the time are *variables;* the principal and the rate are *constants*. The interest *varies directly as* the time. The relation is expressed by the formula

$$i=prt.$$

Dependence of one quantity upon another is of frequent occurrence. For example, the amount a man earns depends upon the number of days that he works; an agent's commission depends upon the amount of goods that he sells; the amount of the electricity used determines the amount of the bill; etc.

In the statement of a problem a letter that may represent different numbers is called a ***variable***, whereas a ***constant*** represents only one number.

EXERCISES

Illustrate direct variation from each of the topics in arithmetic named in Exs. 1–8. Name the variables and the constant. Write a formula using whatever letters you think most suitable.

1. A mechanic is paid 60 cents per hour. Total wages vary ··· (?).

Solution. The mechanic's total wages vary directly as the number of hours that he works. His total wages and the time are variables, the wage per hour is the constant. The relation may be expressed by a formula,

$$W = tw,$$

where W = total wages, t = number of hours, and w = wage per hour.

2. Coal is selling at \$9.50 per ton. Total cost varies ··· (?).

3. The profit is 25% of the cost. Profit varies ··· (?).

4. The profit is 20% of the selling price. Profit varies ··· (?).

5. An agent gets a commission of 5%. Commission varies ··· (?).

6. The tax rate is \$17.50 per thousand valuation. Taxes vary ··· (?).

7. The cost of electricity is 10 cents per kilowatt hour. Total cost varies ··· (?).

8. The Third Liberty Bond pays interest at $4\frac{1}{4}$%. Total income varies ··· (?).

9. A certain quantity of flour costs \$4.75. What will be the cost of m times as much?

10. Knowing how much a plumber will earn in 8 hours, find out how much he will earn in n hours.

11. In a certain city the taxes on a given piece of property are \$48. At the same rate, what will be the taxes on another piece of property valued at b times the first?

12. During a certain month your gas bill is \$4.50. The next month you use $\frac{a}{4}$ as many cubic feet of gas. What is your gas bill?

13. Explain the statement: In making bread the number of one-pound loaves varies directly as the number of pounds of flour used.

14. Explain the statement: For a given state the number of people per square mile varies directly as the total population of the state.

§ 79. Variables in Geometry.

EXERCISES

Exercises 1–4 illustrate direct variation. Name the variables and the constant. Write the formula.

1. The area of a rectangle having a 16″ base varies ⋯ (?).

SOLUTION. The area varies directly as the height. The area and the height are variables, the length is a constant.

FORMULA. $A = 16h$.

2. The area of a triangle having a 10″ altitude varies ⋯ (?).

3. The perimeter of a square varies ⋯ (?).

4. The volume of a cylinder having a base containing 50 sq. in. varies ⋯ (?).

5. The area of a rectangle that has a given height varies ⋯ (?).

6. Explain the statement: The circumference of a circle varies directly with the diameter.

(*a*) Write the formula.

(*b*) Name the constant.

7. The diameter of one circle is three times that of another.

(*a*) What is the ratio of their circumferences?

(*b*) What is the circumference of the larger, if that of the smaller is 15″?

8. Explain the statement: The area of a rectangle *varies jointly as* its base and its height.

SOLUTION. The formula is: $A = bh$.

As b increases, A increases in the same ratio.
As h increases, A increases in the same ratio.
That is, A varies directly as b, also A varies directly as h.
Hence A varies jointly with b and h.

9. A rectangle has an area of 40 sq. in.

(*a*) What would be the area of a rectangle twice as long?

(*b*) What would be the area of a rectangle twice as high?

(*c*) What would be the area of a rectangle both twice as long and twice as high?

10. A rectangular field has an area of 1000 sq. ft. What is the area of a field three times as long and twice as wide?

11. Explain the statement: The volume of a block varies jointly with its length, its width, and its height.

12. A certain box contains 200 cu. in.

(*a*) How much would a box contain that is three times as long?

(*b*) How much would a box contain that is half as high?

(*c*) How much would a box contain that is twice as wide?

(*d*) How much would a box contain that is three times as long, one half as high, and twice as wide?

13. Explain the statement: The area of a circle varies directly as the square of its radius.

SOLUTION. The formula is: $A = \pi r^2$.

As r^2 increases, A increases at the same ratio; that is, A varies directly as r^2. The constant ratio is π.

14. The radius of one circle is twice that of another circle.

(*a*) The square of the radius of the first circle is how many times the square of the radius of the second circle?

(*b*) What is the ratio of the areas?

(*c*) The area of the smaller circle is 25 sq. in. What is the area of the larger circle?

15. Two circles have radii of 2″ and 6″. What is the ratio of their areas?

SOLUTION. ① $\quad \frac{r_1}{r_2} = \frac{}{3}$

② $\quad \frac{r_1^2}{r_2^2} = \frac{1^2}{3^2} = \frac{1}{9} \quad$ ① Squared

③ $\quad \frac{A_1}{A_2} = \frac{1}{9} \quad$ ② Substitution

Ans. $\frac{1}{9}$.

16. Two circles have radii of 5″ and 3″. What is the ratio of their areas?

17. The area of a circle varies with the square of its diameter. The diameter of one circle is 4 times that of another. What is the ratio of their areas?

18. The diameter of one circle is $2\frac{1}{2}$ times that of another. What is the ratio of their areas?

§ 80. Variables in Science.

EXERCISES

1. The distance that a train travels varies directly as the time, given its average rate per hour.

(*a*) Write the formula. Name the variables and the constant.

(*b*) The train travels 145 miles in a certain number of hours. How far will it go in three times as many hours?

2. The distance that a body travels varies jointly as the rate per hour and the number of hours. An airplane goes 140 miles in a certain number of hours.

(*a*) How far will it go in twice as many hours?

(*b*) If the rate is doubled, how far will it go in the given number of hours?

(*c*) How far will it go at twice the first rate and in three times as many hours?

3. The weight of water varies directly as the volume. A certain volume of water weighs 450 lb. What is the weight of three times as much?

4. The specific gravity of a substance varies directly as the density. A cubic foot of steel weighs 490 lb. and a cubic foot of pine weighs 25 lb. What is the ratio of their specific gravities?

5. The distance passed over by a falling body varies directly as the square of the time. What is the ratio of the distance passed over if the time is doubled?

6. The current of electricity (in amperes) varies directly as the electromotive force (in volts). Doubling the voltage will have what effect upon the current?

§ 81. Inverse Variation.

Example. An airplane travels 270 miles in 3 hours, hence its average rate is 90 miles an hour. On another trip it travels 270 miles in 5 hours, hence its average rate is 54 miles an hour. The *greater* the time required the *slower* is the rate. The rate is said to ***vary inversely as*** the time. The formula is: $d = rt$.

Letting d be a constant and giving larger and larger values to t, you note that r becomes smaller and smaller.

EXERCISES

1. With a certain sum of money the number of articles of the same kind that can be purchased varies inversely as the price per article. Explain.

2. With a certain sum of money I can purchase 50 pounds of flour. If the price is doubled, how many pounds can I purchase?

3. If a rectangle has a given area, its length varies inversely as its width. Explain.

4. A rectangle 20′ wide has a certain area. A second rectangle has the same area but is four times as long; how wide is it?

5. The law of the lever states that the *moment* (weight $\times$ weight arm) on one side of the fulcrum equals the *moment* (force $\times$ force arm) on the other side of the fulcrum. For a given *moment*, the force varies inversely as its distance from the fulcrum. To raise a certain weight, a 30-lb. force is required at a certain distance from the fulcrum. How large a force would need to be applied at a distance three times as far from the fulcrum?

6. The volume of gas varies inversely as the pressure upon it. The pressure upon a certain quantity of gas is 40 lb. per square inch. What will be the effect upon the volume if the pressure is reduced to 20 lb. per square inch?

7. The intensity of the current of electricity in a circuit varies directly as the electromotive force and inversely as the resistance. The relation is expressed by the formula: $I=\frac{E}{R}$.

(*a*) What is the effect upon the intensity of doubling the electromotive force?

(*b*) What is the effect of doubling the resistance?

(*c*) The electromotive force of a current is increased threefold and the resistance doubled. If the intensity was 12 amperes before the change, what is it after the change?

8. The intensity of light on an object varies inversely as the square of the distance between the object and the source of light. An object is moved from a distance of 1 foot from a lamp to a distance of 3 feet. What will be the ratio of the intensity of light at the greater distance to the intensity at the nearer distance?

§ **82. Functions.** If two variables are so related that when a value of one is given, a corresponding value of the other is determined, the second variable is called a ***function*** of the first. In the formula $c=\pi d$, c is a *function* of d. In the formula $A=\pi r^2$, A is a *function* of r. In the formula $p=rb$, p is a *function* of b, if r is constant. The total cost of a quantity of sugar is a *function* of the number of pounds, if the price per pound is constant. In the equation $y=3x+2$, y is a function of x. In the equation

$3x+2y=4$, y may be expressed as a function of x, giving $y=\frac{4-3x}{2}$. Or x may be expressed as a function of y, giving $x=\frac{4-2y}{3}$. In the statement $y=\frac{4-3x}{2}$, y is called the *dependent* variable and x is called the *independent* variable.

In the statement $x=\frac{4-2y}{3}$, x is the *dependent* variable and y is the *independent* variable

EXAMPLE. Given the equation $2x-3y=5$, express x as a function of y.

SOLUTION. ① $2x-3y=5$

② $2x=5+3y$ ①$+3y$

③ $x=\frac{5+3y}{2}$ ②$\div 2$

Ans. $x=\frac{5+3y}{2}$.

EXERCISES

In Exs. 1–6, express x as a function of y.

1. $x+y=7$
2. $2x-y=8$
3. $3x+4y=5$
4. $2x-3y=5$
5. $4x+y=7$
6. $3x-2y=0$

In Exs. 7–12, express y as a function of x.

7. $x+y=12$
8. $2x+y=7$
9. $3y-4x=10$
10. $2x-3y=0$
11. $4x-5y=7$
12. $3x-7y=15$

In Exs. 13–18, make x the dependent variable.

13. $x^2+y^2=18$

    Ans. $x=\pm\sqrt{18-y^2}$.
14. $x^2-y^2=40$
15. $5x^2-y^2=15$
16. $xy=15$
17. $3x^2+2y^2=18$
18. $xy+y^2=12$

§ 83. Graphs of Linear Functions of Two Variables. On pages 105–107 you plotted graphs of linear equations containing two variables. In each case, before getting pairs of values, you expressed one variable as a *function* of the other. You noted that, when the equations were of the first degree, the graph was a straight line, hence only two pairs of values were essential for determining its direction. You tabulated three pairs of values, the purpose of the third pair being to check the accuracy of the other two.

In Exercises 1–3 which follow, draw the graphs on the same sheet of squared paper. Make a comparison of these *three* graphs, noting the effect upon the graphs when certain constants are changed.

EXERCISES

1. Given the functions $y=x$, $y=2x$, $y=4x$, and $y=-4x$:

(*a*) Find three pairs of values for each function.

(*b*) Plot the graphs of the four functions on the same axes.

(*c*) Through what point do all the graphs pass?

(*d*) What change in the graph is produced by changing the coefficient of x?

2. Given the functions $y=x$, $y=x+2$, $y=x+4$, and $y=x-2$:

(*a*) Find three pairs of values for each function.

(*b*) Plot the graphs of the four functions on the same axes.

(*c*) What change in the graph is produced by changing the constant?

3. Given the functions $2x+3y=5$, $3x+2y=5$, and $4x+y=5$:

(*a*) Find three pairs of values for each function.

(*b*) Plot the graphs of the three functions on the same axes.

(*c*) What name can you give to these three equations? (See page 106, Ex. 1.)

4. The graph of the function $y=kx$, where k is a constant, passes through what point? [See Ex. 1 (*c*).]

5. The graphs of $y=mx+b$ and $y=mx+c$ are how located, if m has the same value in each? [See Ex. 2 (*c*).]

§ 84. Graphs of Quadratic Functions of Two Variables. Quadratic equations are equations of the second degree. You will see that the graph of an equation of the second degree is a curve, not a straight line; hence you will need to tabulate many pairs of values before plotting a quadratic function.

Example 1. Plot the graph of the function $y=x^2$.

Solution.

x	-5	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$	$+5$
y	25	16	9	4	1	0	1	4	9	16	25

The points are located in Fig. 28 and a smooth curve is drawn through them. Figure 28 is the graph for the squares of numbers from 1 to 10. From the graph, read the squares of 6, -7, $4\frac{1}{2}$, $-5\frac{1}{2}$, $8\frac{1}{2}$.

(Suggestion. Let $x=6, -7, 4\frac{1}{2}$, etc. Then locate, on the curve, the corresponding value of y.)

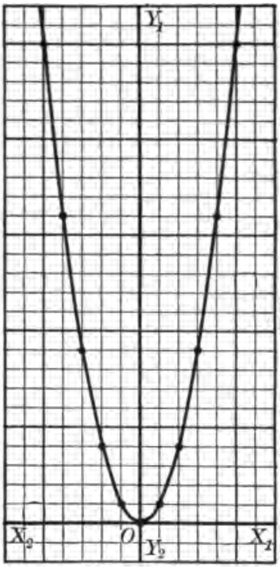

FIG. 28.

EXAMPLE 2. Plot the graph of the function $x^2+y^2=25$.

SOLUTION. Express y as a function of x.

①	$x^2+y^2=25$	
②	$y^2=25-x^2$	① $-x^2$
③	$y=\pm\sqrt{25-x^2}$	② $\sqrt{\ }$

In tabulating values for x and y, values of x are avoided that render the quantity under the radical sign negative.

x	-5	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$	$+5$
y	0	± 3	± 4	± 4.5	± 4.9	± 5	± 4.9	± 4.5	± 4	± 3	0

Figure 29 is the graph of $x^2+y^2=25$, a circle.

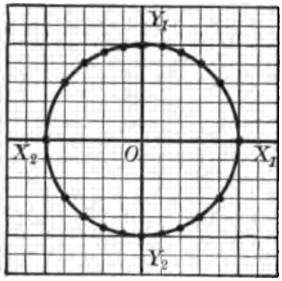

FIG. 29.

EXERCISES

Plot the graphs of each of the following quadratic functions. First express one variable as a function of the other, then tabulate pairs of values.

1. $x^2+y^2=16$. (A circle)
2. $4x^2+y^2=4$. (An ellipse)
3. $y^2=8x+1$. (A parabola)
4. $xy=8$. (An hyperbola)
5. $4x^2-y^2=4$. (An hyperbola)

CHAPTER XIV

FRACTIONS AND EQUATIONS

§ 85. Fractions. In algebra a ***fraction*** is an indicated division, in which the dividend and the divisor are algebraic expressions. The dividend is called the ***numerator*** of the fraction, and the divisor the ***denominator*** of the fraction. The two are called the ***terms*** of the fraction.

The numerator of a fraction may have any value whatever, while the denominator may have any value excepting zero.

The principles of fractions used in arithmetic apply also in algebra.

One important principle is:

The numerator and the denominator of a fraction may be multiplied by, or divided by, the same number without changing the value of the fraction. The resulting fraction is said to be *equivalent* to the given fraction. For example:

(*a*) $\frac{m+n}{x}=\frac{mx+nx}{x^2}$ Both terms of the fraction are multiplied by x.

(*b*) $\frac{5x+5y}{10a}=\frac{x+y}{2a}$ Both terms of the fraction are divided by 5.

(*c*) $\frac{x^2-y^2}{x^2-xy}=\frac{(x+y)(x-y)}{x(x-y)}=\frac{x+y}{x}$ Both terms of the fraction are divided by $x-y$.

§ 86. Addition and Subtraction of Fractions. As in arithmetic, fractions can be added, or subtracted, only when their denominators are the same, hence the first step is to change all fractions into equivalent fractions having the same denominator.

Example 1. Combine $\frac{3}{4a}+\frac{5}{6a}-\frac{2}{3a}$ into a single fraction.

Solution. $\frac{3}{4a}+\frac{5}{6a}-\frac{2}{3a}=\frac{9}{12a}+\frac{10}{12a}-\frac{8}{12a}=\frac{11}{12a}$.

Ans. $\frac{11}{12a}$.

Example 2. Combine $\frac{3}{x}-\frac{2}{x^2}-\frac{4}{x^3}$ into a single fraction.

Solution. $\frac{3}{x}-\frac{2}{x^2}-\frac{4}{x^3}=\frac{3x^2}{x^3}-\frac{2x}{x^3}-\frac{4}{x^3}=\frac{3x^2-2x-4}{x^3}$

Ans. $\frac{3x^2-2x-4}{x^3}$.

EXERCISES

In each of the following exercises, combine the fractions into a single fraction. Reduce the result, when possible.

1. $\frac{3}{8}+\frac{1}{2}-\frac{1}{3}$

2. $\frac{3a}{5}+\frac{a}{3}-\frac{1}{10}$

3. $\frac{x-2}{5}+\frac{x-1}{2}-\frac{x}{1}$

4. $\frac{3}{y}+\frac{2}{5y}-\frac{5}{10y}$

5. $\frac{3}{x^2}+\frac{2}{x}+\frac{1}{2}$

6. $\frac{5}{y}-\frac{2}{x}+\frac{3}{xy}$

7. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

8. $\frac{1}{b^2}-\frac{2}{ab}+\frac{1}{a^2}$

9. $\frac{a+5}{3}-\frac{1}{a}+2$

10. $\frac{x+2}{3}+\frac{x-1}{2}-\frac{1}{6x}$

11. $\frac{1}{a-b}-\frac{1}{a+b}$

Solution.
$$\frac{1}{a-b}-\frac{1}{a+b}=\frac{a+b}{(a+b)(a-b)}-\frac{a-b}{(a+b)(a-b)}$$
$$=\frac{(a+b)-(a-b)}{(a+b)(a-b)}=\frac{2b}{(a+b)(a-b)}$$

Ans. $\frac{2b}{(a+b)(a-b)}$.

NOTE. It is advisable to keep the denominators in each step in the form of factors.

12. $\dfrac{1}{y+1}+\dfrac{1}{y-1}$

13. $\dfrac{2}{x-1}-\dfrac{2}{x+1}$

14. $\dfrac{a}{(a+b)(a-b)}+\dfrac{2}{a+b}$

15. $\dfrac{a+b}{a}-\dfrac{a-b}{b}$

16. $\dfrac{x-2}{x(x+1)}+\dfrac{3}{x}$

17. $\dfrac{a^2}{a(a-2)}-\dfrac{a}{a-2}+\dfrac{a+2}{a}$

18. $\dfrac{1}{(a+1)(a-1)}-\dfrac{2}{(a+1)}$

19. $\dfrac{y+1}{y+2}-\dfrac{y-1}{y-2}$

20. $\dfrac{1}{y^2-4}-\dfrac{1}{y^2+2y}$

SOLUTION.
$$\frac{1}{y^2-4}-\frac{1}{y^2+2y}\equiv\frac{1}{(y-2)(y+2)}-\frac{1}{y(y+2)}$$
$$\equiv\frac{y}{y(y-2)(y+2)}-\frac{y-2}{y(y-2)(y+2)}$$
$$\equiv\frac{y-y+2}{y(y-2)(y+2)}\equiv\frac{2}{y(y-2)(y+2)}$$

Ans. $\dfrac{2}{y(y-2)(y+2)}$.

21. $\dfrac{a}{a-2}-\dfrac{3a}{a^2-2a}$

22. $\dfrac{2x}{x^2-1}-\dfrac{1}{x+1}$

23. $\dfrac{3}{2m-4}-\dfrac{5}{6m-12}$

24. $\dfrac{2}{a-1}-\dfrac{2a}{a^2-1}$

25. $\dfrac{x+1}{1}-\dfrac{x^2}{x-1}$

26. $\dfrac{a}{a^2-b^2}-\dfrac{1}{a+b}+\dfrac{1}{a-b}$

27. $\dfrac{a}{a-b}-\dfrac{b}{a+b}-1$

28. $\dfrac{3}{x-2}-\dfrac{2}{x+3}-\dfrac{x}{x^2+x-6}$

29. $\dfrac{y+1}{(y-1)^2}-\dfrac{2y}{y^2-1}$

30. $\dfrac{m}{m^2-m-12}-\dfrac{2}{m+3}+1$

§ **87. Multiplication of Fractions.** The product of two or more fractions in arithmetic is found by finding the product of the numerators for a new numerator and the product of the denominators for a new denominator. It is customary to reduce the resulting fraction to its simplest form by applying the principle of fractions stated on page 169.

In algebra the procedure is the same and it is also customary to reduce the resulting fraction to its simplest form.

To find the product of two or more fractions:

(1) *Find the product of the numerators for the numerator of the result.*

(2) *Find the product of the denominators for the denominator of the result.*

(3) *Reduce the resulting fraction to its simplest form by applying the principle of fractions stated on page* 169.

EXAMPLE 1. Find the product: $\frac{5a^2}{3b^2}\times\frac{9ab}{20}$.

SOLUTION. $$\frac{\cancel{5}a^2}{\cancel{3}\underset{b}{\cancel{b^2}}}\times\frac{\overset{3}{\cancel{9}}a\cancel{b}}{\underset{4}{\cancel{20}}}=\frac{3a^3}{4b}$$ *Ans.* $\frac{3a^3}{4b}$.

EXAMPLE 2. Find the product: $\frac{x^2-xy}{x^2-y^2}\times\frac{x+y}{x^2}$.

SOLUTION. $$\frac{x^2-xy}{x^2-y^2}\times\frac{x+y}{x^2}=\frac{\cancel{x}(\cancel{x-y})}{(\cancel{x+y})(\cancel{x-y})}\times\frac{\cancel{x+y}}{\underset{x}{\cancel{x^2}}}=\frac{1}{x}$$

Ans. $\frac{1}{x}$.

NOTE. When the division of both terms of a fraction is shown by *crossing out* the common factors, as in Exs. 1 and 2, the process is called *cancellation*.

EXERCISES

In each of the following exercises find the product, and reduce it.

1. $\frac{5}{18}\times\frac{3}{20}\times\frac{3}{4}$
2. $\frac{-4}{15}\times\frac{5}{8}\times\frac{9}{2}$
3. $\frac{5ax}{3b}\times\frac{6ab}{25xy}$
4. $\frac{3x^2}{2y^2}\times\frac{8xy}{9}$
5. $\frac{m}{n}\times\frac{n}{p}\times\frac{p}{m}$
6. $\frac{x^2}{y^2}\times\frac{ay^3}{x^3}\times\frac{x}{ay}$
7. $\frac{12a^2}{5b^2}\times\frac{10ab}{9a^2}\times\frac{3bc}{4a^3}$
8. $\frac{x+y}{x-y}\times\frac{x-y}{x+2y}$
9. $\frac{x(x+1)}{y(x-1)}\times\frac{y^2}{x^2}$
10. $\frac{a}{a+b}\times\frac{a+b}{a}\times\frac{a-b}{a+b}$
11. $\frac{x^2(x+2)}{x}\times\frac{x^2+3x}{x^2-4}$
12. $\frac{a^2-4}{a^2(a-4)}\times\frac{a^2-16}{a^2-2a}$
13. $\frac{9y^2-4}{4m^5}\times\frac{6m^4}{3y-2}$
14. $\frac{x-y}{x+y}\times\frac{x+y}{x-y}$
15. $\frac{a^2+2ab+b^2}{a^2+ab}\times\frac{a}{a-b}$
16. $\frac{x^2+3x}{x^2-3x}\times\frac{x^2-9}{x^2-x-6}$
17. $\frac{a^2-4}{a^2+5a+6}\times\frac{a^2+3a}{a+1}$
18. $\frac{x-y}{x+2y}\times\frac{x^2-4y^2}{x^2-3xy+2y^2}$
19. $\frac{a^2-2a+1}{a^2-a}\times\frac{a+2}{a^2+a-2}$
20. $\frac{n^2+n}{n^2-n}\times\frac{nx-n}{nx+n}\times\frac{x+1}{x-1}$
21. $\frac{a^2+4a}{a+3}\times\frac{a^2+3a}{a+4}$
22. $\frac{(x-y)^2}{(x+y)^2}\times\frac{x^2-y^2}{x^2-xy}$
23. $\frac{4x^2-1}{2x^2+3x+1}\times\frac{x^2+x}{6x^2+x-2}$
24. $\frac{1-x^2}{3x^2-2x-1}\times\frac{3x^2+x}{x+1}$
25. $\frac{a^2-16}{12-a-a^2}\times\frac{a^2+2a-15}{25-a^2}$
26. $\frac{6x^2-13x+6}{15x^2-7x-2}\times\frac{7+34x-5x^2}{2x^2-17x-21}$

§ 88. **Division of Fractions.** To find the quotient of one fraction divided by another fraction:

(1) *Invert the fraction which is the divisor.*

(2) *Find the product of the two fractions after the divisor has been inverted.*

EXAMPLE 1. Find the quotient: $\frac{5a}{9b} \div \frac{2a^2}{3b^2}$.

SOLUTION. $\frac{5a}{9b} \div \frac{2a^2}{3b^2} = \frac{5\cancel{a}}{\underset{3}{\cancel{9b}}} \times \frac{\overset{b}{\cancel{3b^2}}}{\underset{a}{2\cancel{a^2}}} = \frac{5b}{6a}$ *Ans.* $\frac{5b}{6a}$.

EXAMPLE 2. Find the quotient: $\frac{a-1}{a^2+3a+2} \div \frac{a^2-1}{a^2+2a}$.

SOLUTION. $\frac{a-1}{a^2+3a+2} \div \frac{a^2-1}{a^2+2a} = \frac{a-1}{a^2+3a+2} \times \frac{a^2+2a}{a^2-1}$

$$= \frac{\cancel{a-1}}{(\cancel{a+2})(a+1)} \times \frac{a(\cancel{a+2})}{(a+1)(\cancel{a-1})} = \frac{a}{(a+1)^2}$$

Ans. $\frac{a}{(a+1)^2}$.

EXERCISES

In each of the following exercises find the quotient, and reduce it.

1. $\frac{3}{4} \div \frac{2}{3}$
2. $\frac{10x^2y}{a} \div \frac{8x^2}{a^2}$
3. $\left(\frac{x}{y} \times \frac{a}{b}\right) \div \frac{2}{3}$
4. $\frac{x}{y} \div \left(\frac{a}{b} \times \frac{2}{3}\right)$
5. $\frac{a^2-b^2}{a} \div \frac{a^2-2ab+b^2}{ab}$
6. $\frac{a^2+b^2}{a^2-b^2} \div \frac{a}{a^2-ab}$
7. $\frac{x^2-6x+9}{x+2} \div \frac{x^2-9}{x^2+x-2}$
8. $\frac{m+n}{m^2-mn} \div \frac{m^2+mn}{m^2-n^2}$
9. $\frac{b^2-18b+80}{b^2-5b-50} \div \frac{b^2-15b+56}{b^2-7b}$
10. $\frac{y^2-7y+10}{y^2+2y} \div (y^2+2y-35)$

§ 89. Complex Fractions. A *complex fraction* is a fraction having a fraction in its numerator, or in its denominator, or in both its numerator and denominator.

The complex fraction $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ may be written $\frac{a}{b} \div \frac{c}{d}$.

The complex fraction $\dfrac{a+\frac{1}{a}}{a-\frac{1}{a}}$ may be written

$$\left(a+\frac{1}{a}\right) \div \left(a-\frac{1}{a}\right).$$

All complex fractions can be simplified by treating them as exercises in division of fractions. However, it is often shorter to multiply each term of the complex fraction at once by that expression which is the least common multiple of the denominators of the two terms.

EXAMPLE 1. Simplify: $\dfrac{a+\frac{1}{a}}{a-\frac{1}{a}}$.

SOLUTION. $$\frac{a+\frac{1}{a}}{a-\frac{1}{a}} = \frac{a\left(a+\frac{1}{a}\right)}{a\left(a-\frac{1}{a}\right)} = \frac{a^2+1}{a^2-1}$$

Both terms of the complex fraction are multiplied by a.

Ans. $\dfrac{a^2+1}{a^2-1}$.

EXAMPLE 2. Simplify: $\dfrac{\dfrac{x}{1+x}+1}{\dfrac{1}{1+x}+x}$.

SOLUTION.

$$\frac{\dfrac{x}{1+x}+1}{\dfrac{1}{1+x}+x}=\frac{(1+x)\left(\dfrac{x}{1+x}+1\right)}{(1+x)\left(\dfrac{1}{1+x}+x\right)}=\frac{x+1+x}{1+x+x^2}=\frac{1+2x}{1+x+x^2}$$

Both terms of the fraction are multiplied by $1+x$.

Ans. $\dfrac{1+2x}{1+x+x^2}$.

EXERCISES

Simplify each of the following complex fractions.

1. $\dfrac{\frac{2}{3}}{\frac{5}{6}}$.

2. $\dfrac{3\frac{1}{2}}{5\frac{1}{4}}$.

3. $\dfrac{4\frac{3}{8}}{5\frac{1}{4}}$.

4. $\dfrac{7\frac{5}{12}}{5\frac{4}{9}}$.

5. $\dfrac{3}{5\frac{1}{2}}$.

6. $\dfrac{\frac{1}{x}}{\frac{2}{x^2}}$.

7. $\dfrac{1+\dfrac{2}{a}}{1-\dfrac{2}{a}}$.

8. $\dfrac{1+\dfrac{2}{x}}{1-\dfrac{3}{2x}}$.

9. $\dfrac{x}{1-\dfrac{x}{2}}$.

10. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}$.

11. $\dfrac{x+1+\dfrac{2}{x-1}}{x-\dfrac{1}{x-1}}$.

12. $\dfrac{x-4+\dfrac{3}{x}}{1-\dfrac{2}{x}+\dfrac{1}{x^2}}$.

§ 90. Fractional Linear Equations. In every chapter in this course you have been solving equations, many of which contained fractions.

In solving a fractional equation the first step is to get rid of the fractions.

To get rid of the fractions:

(1) *Multiply each member of the equation by the least common multiple* (l. c. m.) *of the denominators.*

(2) *Divide the numerator and denominator of every fraction by all factors common to both.* This process is called ***cancellation.***

EXAMPLE 1. A surveyor drives two stakes in the ground 220 ft. apart. He wishes to drive a third stake so that its distance from the farther stake divided by its distance from the nearer stake is 1.2. Find its distance from each stake.

SOLUTION. Let d = the distance of the third stake from the nearer stake, then

$220-d$ = its distance from the farther stake.

The equation is:

①	$\frac{220-d}{d}=1.2$	
②	$\frac{\cancel{d}(220-d)}{\cancel{d}}=1.2d$	①$\times d$
③	$220=2.2d$	②$+d$
④	$100=d$	③$\div 2.2$
⑤	$220-d=120$	
CHECK.	$\cancel{\frac{120}{100}}=1.2$	

Ans. 120 ft. and 100 ft.

EXAMPLE 2. The denominator of a fraction exceeds its numerator by 2. If 1 is added to both terms of the fraction, the resulting fraction will be equal to $\frac{2}{3}$. Find the fraction.

SOLUTION. Let n = the numerator of the fraction, then

$n+2$ = the denominator, and

$\frac{n}{n+2}$ = the fraction.

The equation is:

$$① \quad \frac{n+1}{n+3}=\frac{2}{3}$$

The l. c. m. of the denominators is $3(n+3)$.

$$② \quad \frac{3\cancel{(n+3)}(n+1)}{\cancel{n+3}}=\frac{\cancel{3}(n+3)\times 2}{\cancel{3}} \qquad ①\times 3(n+3)$$

$$③ \quad 3(n+1)=2(n+3) \qquad ②\equiv \text{(Cancellation)}$$

$$④ \quad 3n+3=2n+6 \qquad ③\equiv \text{(Parentheses removed)}$$

Solving ④, $n=3$

$n+2=5$

CHECK. $\frac{3+1}{5+1}=\frac{2}{3}$ *Ans.* $\frac{3}{5}$.

EXAMPLE 3. Solve the equation $\frac{x-1}{x}-\frac{x-2}{3x}=\frac{1}{2}$.

SOLUTION. ① $\frac{x-1}{x}-\frac{x-2}{3x}=\frac{1}{2}$

The l. c. m. of the denominators is $6x$.

$$② \quad \frac{6\cancel{x}(x-1)}{\cancel{x}}-\frac{\overset{2}{\cancel{6}}\cancel{x}(x-2)}{\cancel{3}\cancel{x}}=\frac{\overset{3}{\cancel{6}}x}{\cancel{2}} \qquad ①\times 6x$$

$$③ \quad 6(x-1)-2(x-2)=3x \qquad ②\equiv \text{(Cancellation)}$$

$$④ \quad 6x-6-2x+4=3x \qquad ③\equiv \text{(Parentheses removed)}$$

Solving ④, $x=2$

CHECK. $\frac{2-1}{2}-\frac{2-2}{3\times 2}\overset{?}{=}\frac{1}{2}$

$\frac{1}{2}-0=\frac{1}{2}$ *Ans.* $x=2$.

NOTE. Equation ③ may be obtained directly from ① by dividing the l. c. m. of the denominators by each denominator and multiplying the corresponding numerator by the quotient.

EXERCISES

Solve each of the following equations, and check each answer.

1. $\frac{3}{x}=1$
2. $2=\frac{3}{m}$
3. $\frac{12}{5a}=-1$
4. $\frac{10}{a}-1=0$
5. $1.8=\frac{81}{2y}$
6. $12-\frac{1}{a}=0$
7. $5+\frac{1}{p}=8$
8. $\frac{9}{x+2}=3$
9. $\frac{2.5}{1-y}=1$
10. $\frac{1.5}{2x-1}=2.7$
11. $\frac{1}{m+6}=\frac{1}{15}$
12. $\frac{1}{2x+1}=\frac{1}{x+7}$
    [l. c. m.: $(2x+1)(x+7)$]
13. $\frac{5}{a-1}=\frac{7}{a+1}$
14. $\frac{7}{x+3}=\frac{6}{x-2}$
15. $\frac{3}{2y-2}=\frac{7.5}{2y-1}$
16. $\frac{7a}{3a+4}=2$
17. $\frac{1}{2x}+\frac{1}{x}=3$
18. $\frac{1}{3x}+\frac{1}{2x}=1$
19. $\frac{1}{R}=\frac{1}{30}+\frac{1}{40}+\frac{1}{15}$
20. $\frac{1}{R}=\frac{1}{4.5}+\frac{1}{1.5}+\frac{1}{13.5}$

21. $\dfrac{3\frac{1}{2}}{1\frac{1}{3}x}=7$

Solution. First multiply the numerator and denominator of the fraction by 6. (See page 175.)

①	$\dfrac{3\frac{1}{2}}{1\frac{1}{3}x}=7$	
②	$\dfrac{6\times 3\frac{1}{2}}{6\times 1\frac{1}{3}x}=7$	①≡
③	$\dfrac{21}{8x}=7$	②≡
Solving ③,	$x=\frac{3}{8}$	

Ans. $x=\frac{3}{8}$.

22. $\dfrac{3}{\frac{2}{3}m}=12$

23. $\dfrac{12\frac{1}{2}}{1\frac{2}{3}p}=4$

24. $\dfrac{3\frac{2}{3}x}{7\frac{1}{2}}=44$

25. $\dfrac{t}{3\frac{1}{8}}=\dfrac{12\frac{1}{4}}{17\frac{1}{2}}$

26. $\dfrac{t}{2\frac{3}{4}}=\dfrac{9\frac{3}{8}}{12}$

27. $\dfrac{\frac{a}{2}+1}{\frac{a}{4}-1}=4$

28. $\dfrac{\frac{5x}{4}+3}{\frac{x}{2}-1}=8$

29. $\dfrac{\frac{2a}{3}-2}{\frac{2a}{3}+1}=2$

30. $\dfrac{3a-\frac{1}{2}}{2+\frac{4a}{3}}=1$

[In Ex. 27, multiply both terms of the fraction by 4;

$\dfrac{2a+4}{a-4}=4$, etc.]

31. $\dfrac{6}{x+2}+\dfrac{2}{x-2}=\dfrac{16}{x^2-4}$

SOLUTION. Before getting the l. c. m. of the denominators, factor the third denominator. Then the l. c. m. is $(x+2)(x-2)$.

① $$\frac{6}{x+2}+\frac{2}{x-2}=\frac{16}{x^2-4}$$

② $$\frac{6\cancel{(x+2)}(x-2)}{\cancel{x+2}}+\frac{2(x+2)\cancel{(x-2)}}{\cancel{x-2}}=\frac{16\cancel{(x+2)}\cancel{(x-2)}}{\cancel{(x+2)}\cancel{(x-2)}}$$ ① × $(x+2)(x-2)$

③ $$6(x-2)+2(x+2)=16$$ ② ≡ (Cancellation)

Solving ③, $$x=3$$

Ans. $x=3$.

(See Note following Example 3, page 179.)

32. $\dfrac{1}{2x+3}+\dfrac{1}{2x-3}=\dfrac{1}{4x^2-9}$

33. $\dfrac{2x+1}{2x-1}-\dfrac{2x-1}{2x+1}=\dfrac{8}{4x^2-1}$

34. $\dfrac{3-m^2}{m^2-1}=\dfrac{4}{m+1}-\dfrac{m+1}{m-1}$

35. $\dfrac{8}{R+3}-\dfrac{4}{R+1}=\dfrac{16}{R^2+4R+3}$

36. $\dfrac{1}{x}+\dfrac{2}{x-1}=\dfrac{8}{x^2-x}$

37. $\dfrac{1}{4K}+\dfrac{1}{3K}+\dfrac{1}{2}=\dfrac{5}{6K}+1$

38. $\dfrac{5}{m+3}-\dfrac{7}{2m+6}=\dfrac{1}{2}-\dfrac{3}{2(m+3)}$

39. $\dfrac{2b-1}{b-1}-\dfrac{3b}{b^2-1}+\dfrac{4}{b+1}=2$

40. $\dfrac{a-4}{2a-5}=\dfrac{6a^2-20a-13}{4a^2-2a-20}-\dfrac{2a-15}{2a+4}$

PROBLEMS

1. The sum of two numbers is 160. The quotient of the larger divided by the smaller is 4. Find the numbers. (Represent the numbers by n and $160-n$.)

2. What number must be added to each term of the fraction $\frac{3}{8}$ to obtain a fraction equal to $\frac{3}{4}$? (The equation is: $\frac{3+n}{8+n}=\frac{3}{4}$.)

3. What number must be subtracted from each term of the fraction $\frac{17}{20}$ to obtain a fraction equal to $\frac{4}{5}$?

4. What number added to both terms of the fraction $\frac{3}{4}$ will double the value of the fraction?

5. Separate 42 into two parts whose ratio is $\frac{3}{4}$.

6. One half of a certain integer is $\frac{1}{5}$ of the sum of the next two consecutive integers. Find the three integers.

7. The denominator of a fraction exceeds its numerator by 5. If 3 is added to both the numerator and denominator, the resulting fraction will be $\frac{2}{3}$. Find the fraction.

8. The difference between two numbers is 60. If the greater is divided by the less, the quotient is 7 and the remainder is 6. Find the numbers.

9. One lot of land contains 1200 sq. ft. more than a second lot. The ratio of their areas is $\frac{3}{4}$. Find their areas.

10. There are two lots of land, the ratio of whose areas is 1.25. The larger lot contains 750 sq. ft. more than the smaller. Find the area of each lot.

11. There are two lots of land, the ratio of whose areas is 0.625. The smaller lot contains 360 sq. ft. less than the larger. Find the area of each lot.

12. A can paint a certain building in 10 days. B can paint it in 8 days. In how many days can they do it working together?

SOLUTION. A can paint $\frac{1}{10}$ of the building in one day.
B can paint $\frac{1}{8}$ of the building in one day.

Let $d=$ the number of days required for both of them working together, then $\frac{1}{d}=$ the fractional part they both can do in one day. The equation is:

$$\frac{1}{10}+\frac{1}{8}=\frac{1}{d}$$

Solving the equation, $d=4\frac{4}{9}$

CHECK. They both can paint $\frac{1}{4\frac{4}{9}}$, or $\frac{9}{40}$, of the building in one day.

$\frac{1}{10}+\frac{1}{8}=\frac{9}{40}$ *Ans.* $4\frac{4}{9}$ days.

13. A can do a certain piece of work in 3 days and B in $4\frac{1}{2}$ days. How long will it take them working together?

14. A can hoe a certain field in $4\frac{1}{2}$ days, B in $5\frac{1}{2}$ days, and C in $5\frac{1}{2}$ days. How long will it take them working together?

15. A and B can paint a building in 12 days. A can paint it in 20 days. How long would it take B to paint it?

16. An oil tank can be filled by one pump in 8 hours, or by another pump in 12 hours. How long will it take to fill the tank if both pumps are working?

17. Two pumps working at the same time can fill an oil tank in $6\frac{1}{2}$ hours. One pump working alone can fill it in $10\frac{1}{2}$ hours. How long will it take the other pump working alone to fill it?

§ 91. Formulas.

1. In a certain electric circuit an electromotive force (E) of 16.0 volts produces a current (C) of 6.4 amperes. Find the number of ohms of resistance (R).

Formula: $$C=\frac{E}{R}$$

2. In a certain electric circuit an electromotive force of 8.5 volts produces a current of 2.5 amperes. The internal resistance (r) is 1.4 ohms. Find the external resistance (R).

Formula: $$C=\frac{E}{R+r}$$

3. Five equal electric cells connected in series form a battery for ringing bells. The internal resistance (r) of each cell is 0.8 ohm, the voltage of each cell is 1.5. Find the external resistance (R) in the circuit, if the current strength is 1 ampere.

Formula: $C=\dfrac{nE}{R+nr}$, where n = the number of cells.

4. Six electric cells, connected in parallel, form a battery for automatic sparks for lighting gas. The internal resistance of each cell is 0.4 ohm and the voltage of each cell is 1.6. Find the external resistance in the circuit, if the current strength is 8 amperes. Formula:

$$C=\frac{E}{R+\frac{r}{n}}$$

5. An electric circuit between two points is divided into two branches, the resistance on one branch (r_1) is 12 ohms, and the resistance on the other branch (r_2) is 8 ohms. Find the total resistance (R) of the divided circuit. Formula: $$\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}$$

6. In the formula for Ex. 5, $r_1 = 17.4$ ohms, $r_2 = 24.3$ ohms, $R = ?$

7. When a circuit is divided into three branches, the formula is:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

Find R, if $r_1 = 6.4$ ohms, $r_2 = 5.8$ ohms, and $r_3 = 7.9$ ohms.

8. A formula used for certain work with mirrors and lenses is:

$$\frac{1}{F} = \frac{1}{D_i} + \frac{1}{D_o}$$

Find D_i, if $F = 30''$ and $D_o = 45''$.

9. Find F, if $D_i = 30$ cm. and $D_o = 15$ cm.

10. Find D_i, if $F = 25$ cm. and $D_o = 15$ cm.

§ 92. Transformation of Formulas.

EXERCISES

Change the subject of the formula in each of the following exercises.

1. $D = \frac{W}{V}$, $W = ?$

2. $D = \frac{W}{V}$, $V = ?$

3. $C = \frac{E}{R}$, $R = ?$

4. $C = \frac{5}{9}(F - 32)$, $F = ?$

5. $C = \dfrac{E}{R + \frac{r}{n}}$, $R = ?$

6. $C = \dfrac{E}{R + \frac{r}{n}}$, $r = ?$

7. $C = \dfrac{nE}{R + nr}$, $E = ?$

8. $C = \dfrac{nE}{R + nr}$, $R = ?$

9. $\frac{D}{360} = \frac{l}{c}$, $c = ?$

10. $\frac{W_1}{W_2} = \frac{d_2}{d_1}$, $d_1 = ?$

11. $\frac{V_1}{V_2} = \frac{P_2}{P_1}$, $V_2 = ?$

12. $s = \frac{n}{2}(a + l)$, $l = ?$

13. $s=\dfrac{rl-a}{r-1}$, $l=?$

14. $s=\dfrac{rl-a}{r-1}$, $a=?$

15. $s=\dfrac{rl-a}{r-1}$, $r=?$

16. $V_1=V_0(1+0.00365t_1)$, $t_1=?$

17. $\dfrac{1}{R}=\dfrac{1}{r_1}+\dfrac{1}{r_2}$, $R=?$

18. $\dfrac{1}{R}=\dfrac{1}{r_1}+\dfrac{1}{r_2}$, $r_2=?$

19. $\dfrac{1}{F}=\dfrac{1}{D_i}+\dfrac{1}{D_o}$, $F=?$

20. $\dfrac{1}{F}=\dfrac{1}{D_i}+\dfrac{1}{D_o}$, $D_i=?$

§ 93. Fractional Linear Pairs.

EXAMPLE. Solve the pair of equations:

(I) $\dfrac{a}{a+1}-\dfrac{2b}{2b-3}=0$

(II) $\dfrac{3}{a-1}+\dfrac{5}{b+2}=0$

SOLUTION.

(I) $\dfrac{a}{a+1}-\dfrac{2b}{2b-3}=0$

② $a(2b-3)-2b(a+1)=0$ (I)×$(a+1)(2b-3)$

③ $2ab-3a-2ab-2b=0$ ②≡

④ $-3a-2b=0$ ③≡

(II) $\dfrac{3}{a-1}+\dfrac{5}{b+2}=0$

⑤ $3(b+2)+5(a-1)=0$ (II)×$(a-1)(b+2)$

⑥ $3b+6+5a-5=0$ ⑤≡

⑦ $5a+3b=-1$ ⑥-1

Solving equations ④ and ⑦, $a=-2$, $b=+3$.

CHECK.

(I) $\dfrac{-2}{-2+1}-\dfrac{2\times3}{2\times3-3}\overset{?}{=}0$

$\dfrac{-2}{-1}-\dfrac{6}{3}\overset{?}{=}0$

$2-2=0$

(II) $\dfrac{3}{-2-1}+\dfrac{5}{3+2}\overset{?}{=}0$

$\dfrac{3}{-3}+\dfrac{5}{5}\overset{?}{=}0$

$-1+1=0$

Ans. $a=-2$, $b=+3$.

EXERCISES

Solve the following pairs of equations, and check the answers.

1. $\begin{cases} \dfrac{x+3}{2}+5y=9 \\ \dfrac{x-2}{3}-\dfrac{y+9}{10}=0 \end{cases}$

2. $\begin{cases} \dfrac{a+b}{2}-\dfrac{a-b}{3}=\dfrac{7}{6} \\ \dfrac{a+b}{3}+\dfrac{a-b}{4}=\dfrac{5}{4} \end{cases}$

3. $\begin{cases} \dfrac{x-2}{x-3}=\dfrac{y+4}{y+3} \\ \dfrac{x+1}{x+2}=\dfrac{y-11}{y-12} \end{cases}$

4. $\begin{cases} \dfrac{p+5}{5}+\dfrac{q-6}{2}=8 \\ \frac{1}{9}(q-4)-\frac{1}{3}(p+12)=-6 \end{cases}$

5. $\begin{cases} 2+4n=5m \\ \dfrac{4}{n+2}=\dfrac{3}{m-1} \end{cases}$

6. $\begin{cases} \dfrac{2a+3b}{1+4b}=1 \\ \dfrac{4a-2b}{a-b}=-2 \end{cases}$

7. $\begin{cases} \dfrac{x}{y}=\dfrac{5}{4} \\ \dfrac{x+6}{y-2}=\dfrac{8}{3} \end{cases}$

8. $\begin{cases} \dfrac{c+1}{5d}=-1 \\ \dfrac{3c+2d}{d+6}=2 \end{cases}$

9. $\begin{cases} \dfrac{a+2}{7}+8-2a=\dfrac{a-b}{4} \\ 2b-\dfrac{3a-2b}{3}=3a+4 \end{cases}$

10. $\begin{cases} \dfrac{1}{x-y}+\dfrac{1}{x+y}=\dfrac{20}{x^2-y^2} \\ \dfrac{x+2}{3}-\dfrac{y-3}{5}=3 \end{cases}$

PROBLEMS

In solving these problems, use two equations involving two unknowns.

1. Find two numbers whose sum is 46, such that if the greater is divided by the less, the quotient is $6\frac{2}{3}$.

2. The value of a fraction is $\frac{5}{9}$. If 8 is added to the numerator and 6 to the denominator, the value of the resulting fraction is $\frac{2}{3}$. Find the fraction.

3. The sum of the terms of a fraction is 11. If $\frac{3}{4}$ is added to the numerator and $\frac{1}{3}$ to the denominator, the value of the resulting fraction is $\frac{9}{20}$. Find the fraction.

4. If a certain number of two digits is divided by the sum of its digits, the quotient is 7. The tens' digit exceeds the units' digit by 3. Find the number.

(SUGGESTION. Represent the digits by t and u, then the number is $10t+u$. Why?)

5. If a certain number of two digits is divided by one less than its tens' digit, the quotient is 19. The tens' digit is 1 less than half the units' digit. Find the number.

§ 94. Fractional Quadratic Equations.

EXAMPLE. Solve the equation $\frac{x+2}{x-7}-\frac{x+5}{x-5}=1.$

SOLUTION.

① $$\frac{x+2}{x-7}-\frac{x+5}{x-5}=1$$

② $$(x+2)(x-5)-(x-7)(x+5)=(x-7)(x-5)$$
①×$(x-7)(x-5)$

③ $$x^2-3x-10-x^2+2x+35=x^2-12x+35$$
②≡(Parentheses removed)

④ $$-x+25=x^2-12x+35$$ ③≡

⑤ $$0=x^2-11x+10$$
④$+x-25$

Solving equation ⑤ by factoring, $x=1$, or 10.

CHECK. $\frac{1+2}{1-7}-\frac{1+5}{1-5}\stackrel{?}{=}1$ $\qquad$ $\frac{10+2}{10-7}-\frac{10+5}{10-5}\stackrel{?}{=}1$

$\frac{3}{-6}-\frac{6}{-4}\stackrel{?}{=}1$ $\qquad$ $\frac{12}{3}-\frac{15}{5}\stackrel{?}{=}1$

$-\frac{1}{2}+\frac{3}{2}=1$ $\qquad$ $4-3=1$

Ans. $x=1$, or 10.

EXERCISES

Solve each of the following equations, and check both of the roots.

1. $x=\dfrac{8}{x-2}$

2. $b=\dfrac{2}{b}-1$

3. $w-\dfrac{20}{w-5}=6$

4. $\dfrac{6}{x+1}+\dfrac{2}{x}=3$

5. $\dfrac{a^2}{a-2}=\dfrac{4}{a-2}+5$

6. $\dfrac{4x-3}{1-x}=3+\dfrac{x^2}{1-x}$

7. $\dfrac{x+3}{4}=\dfrac{4}{x-3}$

8. $\dfrac{y}{y-1}=\dfrac{5}{6}+\dfrac{y-1}{y}$

9. $\dfrac{6}{a-1}=\dfrac{12}{a^2-1}+1$

10. $\dfrac{2}{x+3}+\dfrac{12}{x^2-9}-1=0$

11. $\dfrac{y}{y-1}+\dfrac{y-3}{y^2-1}=\dfrac{5}{4}$

12. $\dfrac{m}{m+1}+\dfrac{m+1}{m}=2\frac{1}{6}$

PROBLEMS

1. The sum of a number and its reciprocal is $3\frac{1}{3}$. Find the number.

The *reciprocal* of 5 is $\frac{1}{5}$.

The *reciprocal* of $3\frac{1}{2}$ is $\dfrac{1}{3\frac{1}{2}}$, or $\frac{2}{7}$.

The *reciprocal* of a number n is $\dfrac{1}{n}$.

The ***reciprocal*** of a number is 1 divided by that number.

The equation is: $n+\dfrac{1}{n}=3\frac{1}{3}$.

2. A number is 4.8 larger than its reciprocal. Find the number.

3. A number exceeds its reciprocal by 2.1. Find the number.

4. Two thirds of a number is $3\frac{1}{3}$ more than twice its reciprocal. Find the number.

5. The ratio of the square of the smaller of two consecutive integers to the square of the larger is $\frac{9}{16}$. Find the integers.

6. One half the larger of two consecutive integers added to the reciprocal of the smaller is $3\frac{2}{3}$. Find the integers.

7. The ratio of two sides of a rectangle is $\frac{3}{4}$, and its diagonal is 15″. Find the sides.

8. The ratio of the two sides of a rectangle is $\frac{5}{12}$, and its diagonal is 2.6″. Find the sides.

9. The sum of two numbers is 20. The sum of the first number and the reciprocal of the second is $16\frac{1}{4}$. Find the numbers.

10. The numerator of a fraction exceeds its denominator by 2. The sum of the fraction and its reciprocal is $2\frac{4}{15}$. Find the fraction.

11. Solve the formula $s=\dfrac{n(n-1)}{2}$ for n.

12. Solve the formula $\dfrac{m}{p}=\dfrac{p}{c-m}$ for m.

PART II. GEOMETRY

CHAPTER XV

LINES AND ANGLES

§ **95. Straight Lines.** In the measurement and construction of geometric figures you have used the straightedge (ruler), protractor, and compasses. The straightedge was used for drawing straight lines, the protractor for measuring and drawing angles, the compasses for drawing arcs of circles.

From the work with the straightedge, the following properties of straight lines may be accepted as true:

(1) *Only one straight line can be drawn through two points.*

(2) *Two straight lines can intersect in only one point.*

(3) *A straight line is the shortest line between two points.*

These statements are three of the ***axioms*** of geometry. The axioms of geometry are sometimes called ***postulates***.

§ **96. Angles.** An **angle** ($\angle$) is formed when two lines meet at a point. Its size is determined by the amount of rotation of a line in a plane about one of its points from one position to another. The greater the amount of turning the greater the angle.

In Fig. 30, the line is turned about O from the position OA to the position OB, forming the angle AOB. O is called the ***vertex*** of the angle AOB, and OA and OB the ***sides***. In reading an angle the vertex letter is always read in the middle ($\angle AOB$).

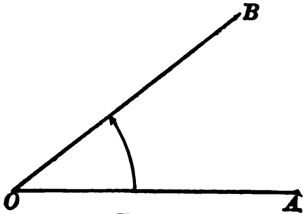

FIG. 30.

If the line (Fig. 30) is turned about O from the position OA until it again takes the position OA, a ***revolution*** is completed. A revolution is divided into 360 equal parts called ***degrees*** (360°). Thus one degree is one three hundred sixtieth of one revolution. The degree is subdivided into 60 equal parts, called *minutes* (′). The minute is divided into 60 equal parts, called *seconds* (″). An angle of 25 degrees, 36 minutes, and 12 seconds is written 25° 36′ 12″.

A ***right angle*** is one fourth of a revolution, or an angle of 90°.

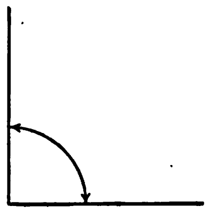

FIG. 31.

An ***acute*** angle is an angle less than 90°.

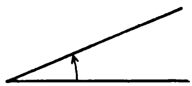

FIG. 32.

An ***obtuse*** angle is an angle more than 90° and less than 180°.

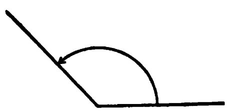

FIG. 33.

The ***bisector*** of an angle is the line that divides the angle into two equal angles.

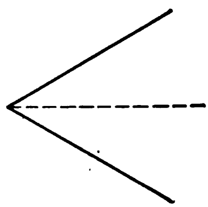

FIG. 34.

CONSTRUCTIONS

1. *To construct a given angle at a given point on a given line.*

Carry out this construction as indicated in Fig. 35, using ruler and compasses. (See Course II, page 100.)

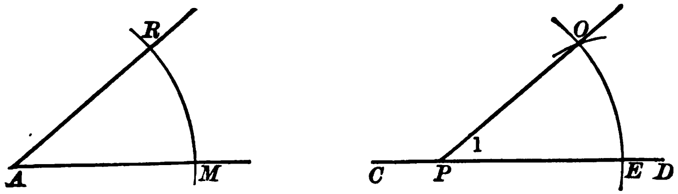

FIG. 35.

2. *To bisect a given angle.* (See Course II, page 101.)

Carry out this construction as indicated in Fig. 36.

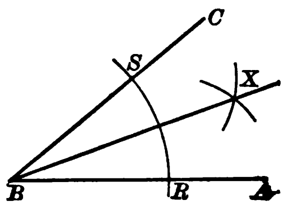

FIG. 36.

§ 97. **Perpendicular Lines.** ***Perpendicular lines*** are lines that form right angles with each other.

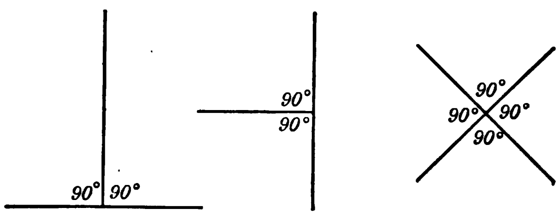

FIG. 37.

The ***perpendicular bisector*** of a given line is the line that is perpendicular to the given line at its mid-point.

CONSTRUCTIONS

1. *To construct the perpendicular bisector of a given line.*

Carry out this construction as indicated in Fig. 38. (See Course II, page 97.)

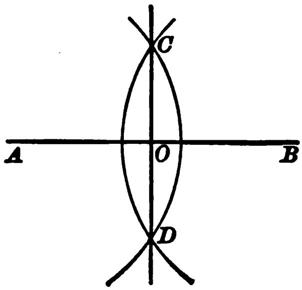

FIG. 38.

2. *To construct a perpendicular to a given line through a given point.*

Carry out these constructions as indicated in Figs. 39 and 40. (See Course II, page 98.)

A. When the given point is on the line.

B. When the given point is not on the line.

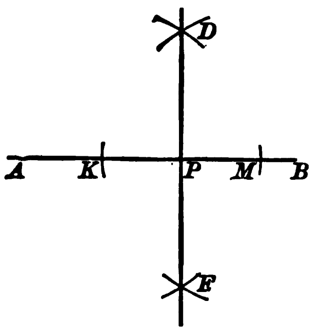

FIG. 39.

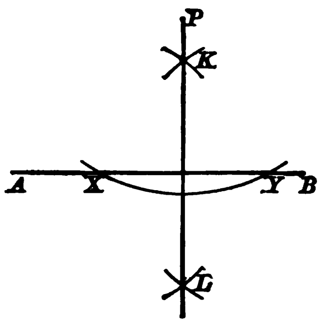

FIG. 40.

FACTS CONCERNING PERPENDICULAR LINES

(1) *All right angles are equal.*

(2) *Through a given point only one perpendicular can be drawn to a given line.*

§ 98. Complementary Angles. *Complementary angles* are two angles whose sum is 90°. Either angle is called the *complement* of the other. In Fig. 41,

① $\angle x+43=90$

② $\angle y+43=90$

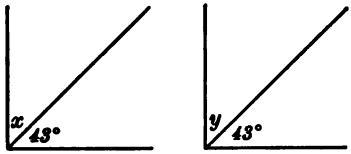

Fig. 41.

③ $\angle x+43=\angle y+43$ (Axiom 5. A quantity may be substituted for its equal in an equation. See page 199.)

④ $\angle x=\angle y$ ③-43 (Axiom 2, page 199)

In ①, $\angle x$ is the complement of 43°; in ②, $\angle y$ is the complement of 43°.

The complements of equal angles are equal.

§ 99. Supplementary Angles. *Supplementary angles* are two angles whose sum is 180°. Either angle is called the *supplement* of the other. In Fig. 42,

① $\angle x+115=180$

② $\angle y+115=180$

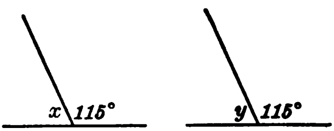

Fig. 42.

③ $\angle x+115=\angle y+115$ (Axiom 5)

④ $\angle x=\angle y$ ③-115 (Axiom 2)

In ①, $\angle x$ is the supplement of 115°; in ②, $\angle y$ is the supplement of 115°.

The supplements of equal angles are equal.

EXERCISES

1. What is the complement of an angle of 22°? Of 39°? Of 57°? Of 86°? Of 58° 24′?

2. What is the supplement of an angle of 34°? Of 69°? Of 101°? Of 128°? Of 154° 6′?

3. Construct the complement of an angle of 75° with ruler and compasses. Check your construction with a protractor.

4. Construct the supplement of an angle of 135° with ruler and compasses. Check your construction with a protractor.

5. Two angles are complementary and the greater exceeds the smaller by 18°. Find each angle.

6. Two angles are complementary and the greater is twice the smaller. Find each angle.

7. The angles x and y are complementary and $x=3y$. Find each angle.

8. Two angles are supplementary and the greater exceeds the smaller by 52°. Find each angle.

9. The angles x and y are supplementary and $x=2y$. Find each angle.

10. The ratio of two complementary angles is $\frac{2}{3}$. Find each angle.

11. The ratio of two complementary angles is $\frac{3}{7}$. Find each angle.

12. The ratio of two supplementary angles is $\frac{5}{7}$. Find each angle.

13. Two angles are complementary and one is 6° more than twice the other. Find each angle.

14. Two angles are complementary and one is 18° less than five times the other. Find each angle.

§ 100. Vertical Angles. When the sides of one angle are the prolongations of the sides of another angle, these angles are called ***vertical angles***.

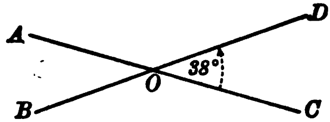

FIG. 43.

Thus in Fig. 43, $\angle AOB$ and $\angle COD$ are vertical angles; also $\angle BOC$ and $\angle DOA$ are vertical angles. To construct this figure, first draw the line AC. Then, using your protractor, draw BD, making $\angle COD$ equal to 38°.

How many degrees are there in $\angle AOD$?

How many degrees are there in $\angle AOB$?

How many degrees are there in $\angle BOC$?

How do the angles COD and AOB compare?

How do the angles BOC and AOD compare?

If one straight line intersects another straight line, the vertical angles formed are equal.

§ 101. Adjacent Angles. Two angles which have the same vertex and a common side between them are called ***adjacent angles***. Thus in Fig. 44, $\angle AOB$ and $\angle BOC$ are adjacent angles, for they have the same vertex O, and the common side OB between them.

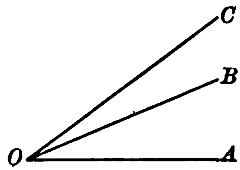

FIG. 44.

EXERCISES

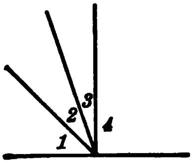

FIG. 45.

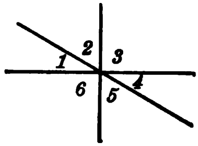

FIG. 46.

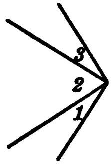

FIG. 47.

1. In Figs. 45–47, read the pairs of angles that are adjacent. Read the vertical angles. Read the angles that are complementary. Read the angles that are supplementary. Read the angles that are both adjacent and complementary at the same time. Read the angles that are both supplementary and adjacent at the same time.

2. In Fig. 48, $\angle AOD = 90°$. What is the value of each of the other angles? Explain.

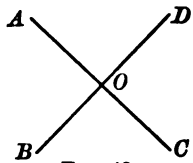

FIG. 48.

3. In Fig. 49, $\angle m = 35°$. Express the values of $\angle n$, $\angle x$, and $\angle y$.

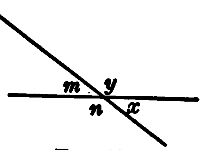

FIG. 49.

4. In Fig. 50, the three straight lines cross at O. If $\angle a = 42°$ and $\angle a + \angle f = 90°$, find the value of each of the other angles.

5. In the figure of Ex. 4, if $\angle a$ were 26° and $\angle b$ were 90°, find the value of each of the other angles.

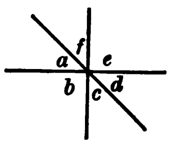

FIG. 50.

6. The angle of elevation of the sun above the horizon, as $\angle SOA$, may be measured in the following way (Fig. 51).

Hold a quadrant in a vertical position, so that a plumb line OP, which is suspended from a pin at O, will fall upon the 90° mark. The shadow of the pin will then fall upon the scale at C. Angle BOC shows the angle of elevation of the sun. Explain.

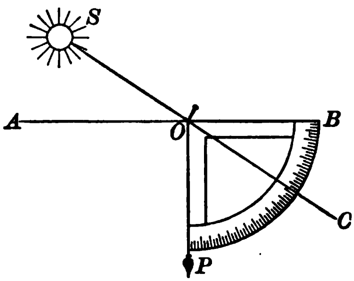

FIG. 51.

§ 102. List of Axioms.

(1) *If the same quantity is added to equal quantities, the sums are equal.*

(2) *If the same quantity is subtracted from equal quantities, the remainders are equal.*

(3) *If equal quantities are multiplied by the same quantity, the products are equal.*

(4) *If equal quantities are divided by the same quantity, the quotients are equal.*

(5) *A quantity may be substituted for its equal in a given equation.*

§ 103. Symbols.

$=$, equals, or is equal to	$\perp$, perpendicular, or is perpendicular to
$\neq$, not equal, or is not equal to	$\angle$, angle
$\equiv$, identical, or is identical to	$\triangle$, triangle
$\sim$, similar, or is similar to	$\square$, parallelogram
$\cong$, congruent, or is congruent to	$\odot$, circle
$\parallel$, parallel, or is parallel to	

The plural of any symbol representing a noun is obtained by using the letter *s*. Thus, $\measuredangle$ for angles; $\blacktriangle$ for triangles.

CHAPTER XVI

CONGRUENT TRIANGLES

§ 104. Classification of Triangles. A ***triangle*** (△) is a plane figure inclosed by three straight lines.

A. Triangles (▲) are divided into three groups according to the lengths of their sides:

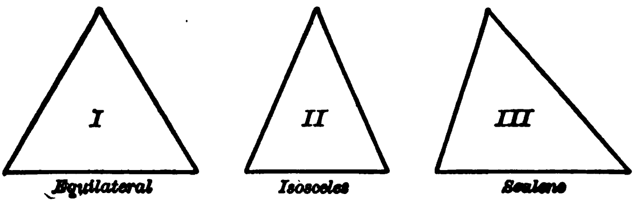

Fig. 52.

(*a*) ***Equilateral*** triangles, having all three sides equal.

(*b*) ***Isosceles*** triangles, having two sides equal.

(*c*) ***Scalene*** triangles, having no two sides equal.

B. Triangles are divided into three groups according to their angles:

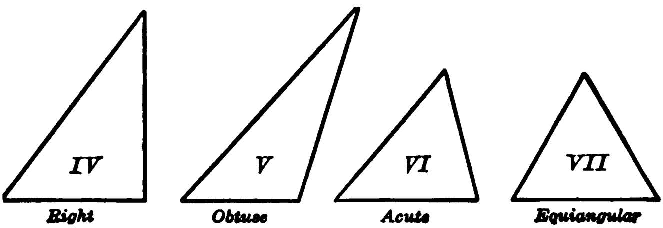

Fig. 53.

(*a*) ***Right*** triangles, having one angle right (90°).

(*b*) ***Obtuse*** triangles, having one angle obtuse.

(*c*) ***Acute*** triangles, having all three angles acute.

In an acute triangle, if all three angles are equal, the triangle is called ***equiangular***.

§ 105. Corresponding Angles and Sides. If two triangles have the angles of the one respectively equal to the angles of the other, the equal angles are called ***corresponding angles***, and the sides opposite these angles are called ***corresponding sides***.

§ 106. Congruent Triangles. ***Congruent triangles*** are triangles that can be made to coincide in all their parts.

The symbol, ≅, means "congruent," or "is congruent to," page 199.

§ 107. Motion. When we said that the size of an angle was determined by the amount of rotation of a line in a plane about one of its points, the idea of motion was implied; when we used the compasses in the construction of angles, perpendiculars, etc., the idea of motion was also implied.

For the study of geometry it is necessary that you get a clear idea of the three kinds of motion that are possible:

(1) *Any figure in a plane may be imagined to slide along the plane from one position to another.*

(2) *Any figure in a plane may be imagined to rotate in the plane about any one of its points.*

(3) *Any figure in a plane may be imagined to make a complete revolution about any line of the plane as an axis until it comes into the plane again.* This motion is sometimes called *overturning*.

These three statements are the *postulates* of *motion*.

§ **108.** THEOREM I.* ***If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the two triangles are congruent.***

On a piece of tracing paper draw a straight line; on this line mark off with the compasses a line-segment $XY = 1.6''$.

At X draw an angle of 35°, and on the other side of $\angle x$ mark off with the compasses $XZ = 1.3''$.

Complete the $\triangle XYZ$.

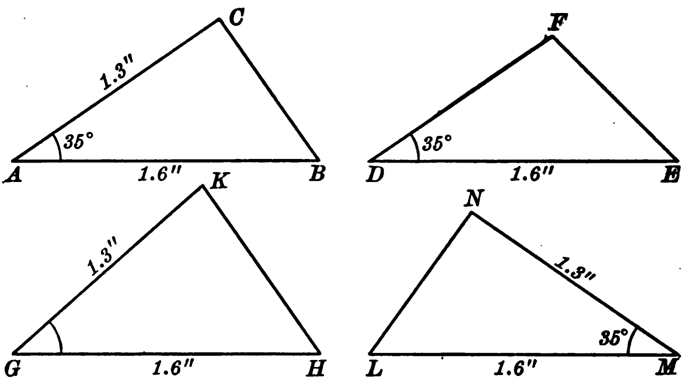

FIG. 54.

(*a*) Comparison of $\triangle XYZ$ with $\triangle ABC$.

Slide $\triangle XYZ$ along the page until point X falls on point A.

Rotate $\triangle XYZ$ about A until XY lies along AB.

Does point Y then fall on point B? $(XY = AB)$

Does XZ then lie along AC? $(\angle X = \angle A)$

Does point Z then fall on point C? $(XZ = AC)$

Do the $\triangle$s XYZ and ABC coincide; that is, do they fit exactly in all their parts?

* A *theorem* is a geometric statement requiring proof.

Since the $\triangle XYZ$ and ABC are precisely the same shape and size, they are said to be *congruent;* that is,

$$\triangle XYZ \cong \triangle ABC$$

(*b*) Comparison of $\triangle XYZ$ with $\triangle DEF$.

Slide $\triangle XYZ$ along the page until point X falls on point D.

Rotate $\triangle XYZ$ about D until XY lies along DE.

Does point Y then fall on point E? ($XY = DE$)

Does XZ then lie along DF? ($\angle X = \angle D$)

Does point Z then fall on point F? ($XZ \neq DF$)

Is $\triangle XYZ \cong \triangle DEF$?

NOTE. The symbol, $\neq$, means "is not equal to," page 199.

(*c*) Comparison of $\triangle XYZ$ with $\triangle GHK$.

Slide $\triangle XYZ$ along the page until point X falls on point G.

Rotate $\triangle XYZ$ about G until XY lies along GH.

Does point Y then fall on point H? ($XY = GH$)

Does XZ then lie along GK? ($\angle X \neq \angle G$)

Is $\triangle XYZ \cong \triangle GHK$?

(*d*) Comparison of $\triangle XYZ$ with $\triangle LMN$.

Slide $\triangle XYZ$ along the page until point X falls on point M.

Rotate $\triangle XYZ$ about M until XY lies along ML.

Does point Y then fall on point L? Why?

(Note that the two $\triangle$ are now on opposite sides of ML.)

Overturn $\triangle XYZ$ on ML as an axis, bringing $\triangle XYZ$ again into the plane of $\triangle LMN$ and on the same side of LM.

Does XZ then lie along MN? Why?

Does point Z then fall on point N? Why?

Is $\triangle XYZ \cong \triangle LMN$?

After superposing the $\triangle XYZ$ on each of the $\triangle ABC$, DEF, GHK, and LMN, what conclusion do you reach?

State this conclusion as definitely as possible.

In the two $\triangle ABC$ and LMN the equal parts are said to be arranged in the *opposite order*.

§ 109. **Proof by Superposition** is the method of proving the congruence of two figures by making them coincide. This method of proof is used in fundamental propositions only. In order to use it, there must be at least one pair of equal angles in the figures being compared. You should always begin by placing a line or angle on its equal part; then, by successive steps, trace the position of the rest of the figure.

NOTE. *In congruent triangles, the corresponding sides are equal and the corresponding angles are equal.*

EXERCISES

1. In Fig. 55, $ABCD$ is a square; E is the mid-point of the side AD.

Can you prove that the triangles ABE and ECD are congruent?

Does this prove that $EC = EB$? Why?

What kind of a triangle is EBC?

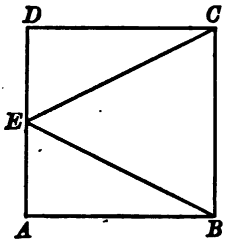

FIG. 55.

2. In Fig. 56, $ABCD$ is a rectangle; $AD = BC$; $AE = FB$.

Can you prove that rt. $\triangle DAE \cong$ rt. $\triangle FBC$?

Can you prove that $ED = FC$?

Can you also prove that $\angle 1 = \angle 2$?

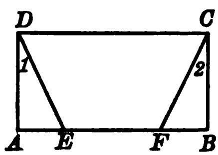

FIG. 56.

3. In Fig. 57, $ABCD$ is a rectangle; $AE=FB$; $DM=MC$.

Can you prove that $\triangle DEM \cong \triangle CFM$ after what was proved in Ex. 2?

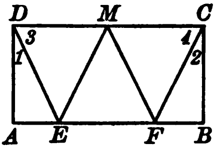

FIG. 57.

(SUGGESTION. $\angle 1 + \angle 3 = \angle 4 + \angle 2$.)

Does $EM = FM$? Why?

What kind of a $\triangle$ is EMF?

4. In Fig. 58, $AO=OB$, and $CO=OD$.

Prove that $AC=BD$.

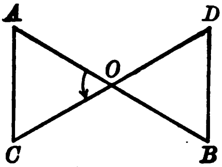

FIG. 58.

5. In Fig. 59, $AB=AC$; $\angle 1 = \angle 2$.

Can you prove that $\triangle ABD \cong \triangle ADC$?

Does this prove that $\angle B = \angle C$?

Does this prove that $BD=DC$?

Does it follow that *the bisector of the vertex angle of an isosceles triangle bisects the base?*

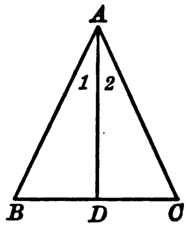

FIG. 59.

NOTE. The *vertex angle* (also called vertical angle) of a triangle is the angle opposite the base. The *base* of a triangle is the side on which the figure is supposed to rest.

6. In Fig. 60, AD is $\perp BC$, and $BE=EC$; A is any point in AD, and AC and AB are drawn.

Prove that $AB=AC$.

Does it follow that *any point in the perpendicular bisector of a line is equidistant from the extremities of the line?*

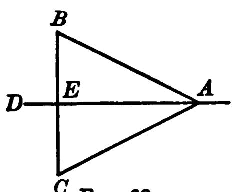

FIG. 60.

7. If two lines bisect each other at right angles, prove that any point in either is equidistant from the ends of the other.

8. In a square $ABCD$, the points V, W, X, Y are the mid-points of the consecutive sides AB, BC, CD, DA, respectively. Prove that $VW = WX = XY = YV$.

9. In Fig. 61, $ABCD$ is a rectangle, and $AF = BE$.

Can you prove that the $\triangle AFD$ and EBC are congruent?

Does this prove that $DF = EC$?

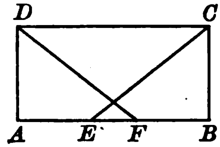

FIG. 61.

10. If D and E, C and F, in Fig. 61, are connected by two straight lines, prove that $\triangle DAE \cong \triangle CBF$.

Does this prove that $DE = CF$?

11. After DE and CF are drawn, in Fig. 61 (Ex. 10), prove that $\triangle DEF \cong \triangle CFE$. (SUGGESTION. $\angle DEF$ is a supplement of $\angle DEA$; $\angle CFE$ is a supplement of $\angle BFC$; then apply § 99.)

Does this also prove that $DF = CE$?

12. In Fig. 62, $ABCD$ is a rectangle.

Prove that $\triangle DAB \cong \triangle CAB$.

Does this prove that $AC = DB$?

Does this prove that the diagonals of a rectangle are equal?

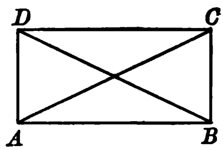

FIG. 62.

NOTE. The *diagonals* of a rectangle are the lines joining the opposite vertices.

13. In Fig. 62, prove the following:

(1) that $\triangle ACD \cong \triangle BDC$,

(2) that $\triangle BAD \cong \triangle CDA$,

(3) that $\triangle ABC \cong \triangle DCB$.

14. In Fig. 63, to measure the distance from A to P, draw $AB \perp$ to AP; make $OB = OA$, and $CB \perp$ to OB at B; then draw OP and extend it to meet CB at C.

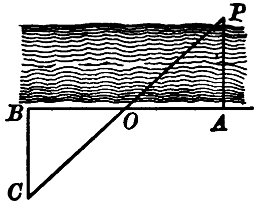

FIG. 63.

Prove that $CB = PA$.

How could the distance AP be found?

15. In Fig. 64, to measure the distance from M to N, measure AN and produce it through A so that $AB = AN$; measure AM and produce it so that $AC = AM$.

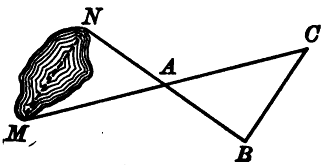

FIG. 64.

Prove that $BC = MN$.

How could you find the distance MN?

16. If equal segments, AB and AC, measured from the vertex of the angle A, are laid off on the sides of the angle, and if B and C are joined to any point M in the bisector of the angle, prove that $BM = CM$.

NOTE. A *segment* is a limited portion of a straight line.

17. If equal segments measured from the vertex are laid off on the equal sides of an isosceles triangle, prove that the lines joining the ends of these segments to the opposite ends of the base are equal.

18. If the equal sides of the isosceles triangle are extended through the vertex and the equal segments are laid off on the sides extended, give the proof.

§ 110. THEOREM II. ***If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the two triangles are congruent.***

On a piece of tracing paper draw a straight line; on this line mark off with the compasses a line-segment $XY = 1.6''$.

At X draw an angle of 35°, and at Y an angle of 60°.

Complete the $\triangle XYZ$.

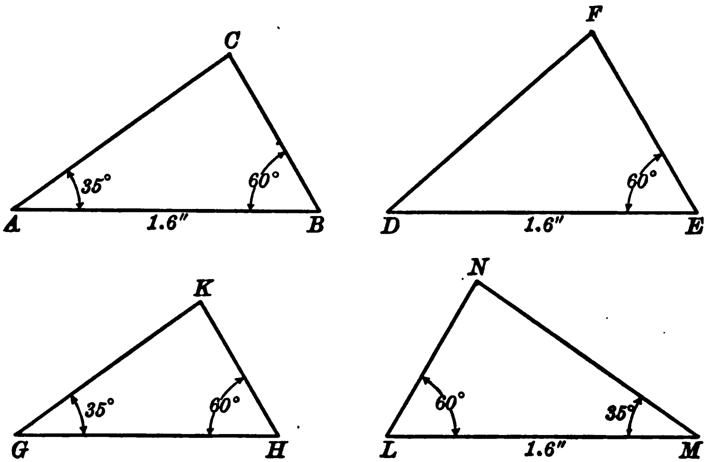

FIG. 65.

(*a*) Comparison of $\triangle XYZ$ with $\triangle ABC$.

Slide $\triangle XYZ$ along the page until point X falls on point A.

Rotate $\triangle XYZ$ about A until XY lies along AB.

Does point Y then fall on point B? Why?

Does XZ then lie along AC? Why?

Does YZ then lie along BC? Why?

Does point Z then fall on point C? Why?

Do the $\triangle XYZ$ and ABC fit exactly in all their parts?

Make a statement about the $\triangle$ XYZ and ABC.

(*b*) Comparison of $\triangle XYZ$ with $\triangle DEF$.

Slide $\triangle XYZ$ along the page until point X falls on point D.

Rotate $\triangle XYZ$ about D until XY lies along DE.

Does point Y then fall on point E? Why?

Does YZ then lie along EF? Why?

Does XZ then lie along DF? $(\angle X \neq \angle D)$

Is $\triangle XYZ \cong \triangle DEF$?

(*c*) Comparison of $\triangle XYZ$ with $\triangle GHK$.

Slide $\triangle XYZ$ along the page until point X falls on point G.

Rotate $\triangle XYZ$ about G until XY lies along GH.

Does XZ then lie along GK? Why?

Does point Y then fall on point H? $(XY \neq GH)$

Is $\triangle XYZ \cong \triangle GHK$?

(*d*) Comparison of $\triangle XYZ$ with $\triangle LMN$.

Slide $\triangle XYZ$ along the page until point X falls on point M.

Rotate $\triangle XYZ$ about M until XY lies along ML. (Note that the two $\triangle$ are now on opposite sides of ML.)

Does point Y then fall on point L? Why?

Overturn $\triangle XYZ$ on ML as an axis bringing $\triangle XYZ$ again into the plane of $\triangle LMN$ and on the same side of LM.

Does XZ then lie along MN? Why?

Does YZ then lie along LN? Why?

Is $\triangle XYZ \cong \triangle LMN$?

After superposing the $\triangle XYZ$ on each of the $\triangle ABC$, DEF, GHK, and LMN, what conclusion do you reach?

P

EXERCISES

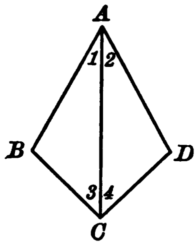

FIG. 66.

1. In Fig. 66, $\angle 1 = 30°$, $\angle 2 = 30°$, $\angle 3 = 45°$, $\angle 4 = 45°$. Prove that $\triangle ABC \cong \triangle ACD$.

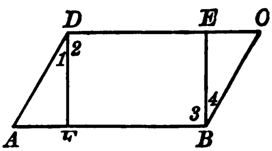

FIG. 67.

2. In the parallelogram $ABCD$ (Fig. 67), $AD = BC$, $\angle A = \angle C$, $\angle 1 = \angle 4$. Prove that $\triangle ADF \cong \triangle BEC$.

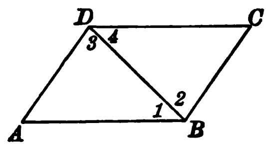

FIG. 68.

3. In the parallelogram $BCDA$ (Fig. 68), $\angle 1 = \angle 4$, $\angle 2 = \angle 3$. Prove that $AB = DC$ and that $AD = BC$.

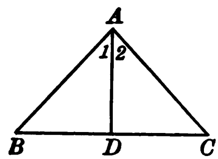

FIG. 69.

4. In the triangle ABC (Fig. 69), $AB = AC$ and $\angle 1 = \angle 2$. Prove that $\triangle ABD \cong \triangle ADC$.

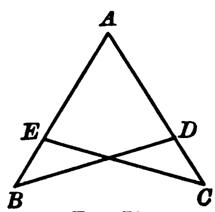

FIG. 70.

5. In Fig. 70, $AB = AC$, $\angle B = \angle C$. Prove that $\triangle ABD \cong \triangle AEC$.

6. To measure the distance (BD) that a boat (B) is anchored from the shore at D (Fig. 71), a boy proceeds as follows. He marks off the line DE, and by sighting to the boat from D and E obtains the $\angle s$ EDB and DEB. He then makes $\angle CDE = \angle EDB$, $\angle DEC = \angle DEB$, and completes the $\triangle DCE$.

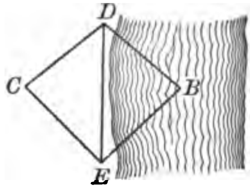

FIG. 71.

Where can he measure the distance equal to BD? Explain.

§ **111**. THEOREM III. *If two sides of a triangle are equal, the angles opposite the equal sides are equal.*

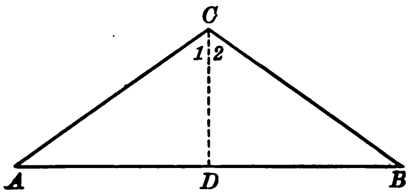

FIG. 72.

In $\triangle ABC$ draw CD so that it bisects $\angle C$.

Compare the two triangles thus formed; that is, $\triangle s$ ACD and CDB.

$AC = BC$	Why?
$\angle 1 = \angle 2$	Why?
$CD \equiv CD$	
$\triangle ACD \cong \triangle CDB$	Why?
$\angle A = \angle B$	Why?

State your conclusion.

EXERCISES

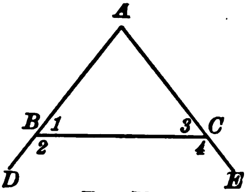

FIG. 73.

1. In Fig. 73, $AB = AC$, and AB and AC are extended to D and E respectively. Prove that $\angle 2 = \angle 4$.

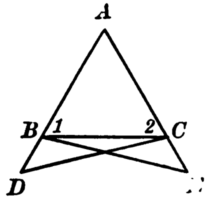

FIG. 74.

2. In Fig. 74, $\angle 1 = \angle 2$, and $BD = CE$. Prove that $\triangle BDC \cong \triangle BEC$.

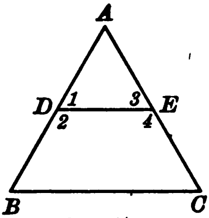

FIG. 75.

3. In Fig. 75, $AB = AC$; $AD = \frac{1}{2}AB$; $AE = \frac{1}{2}AC$. Prove that $\angle 1 = \angle 3$; also $\angle 2 = \angle 4$.

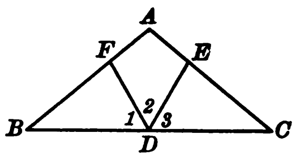

FIG. 76.

4. The triangle ABC (Fig. 76) is an isosceles triangle; D is the mid-point of BC; $\angle 1 = \angle 3$. Prove that $FD = ED$.

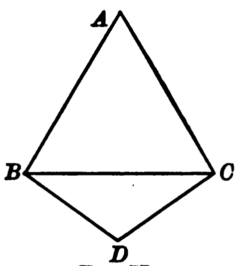

FIG. 77.

5. The triangles ABC and DBC (Fig. 77) are isosceles triangles. Prove that $\angle ABD = \angle ACD$.

6. In Fig. 78, $AB=AD$, and $BC=CD$; BD is drawn. Prove that $\angle ABC = \angle ADC$.

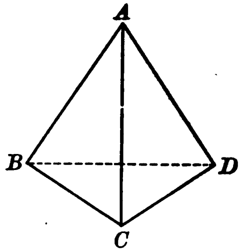

Fig. 78.

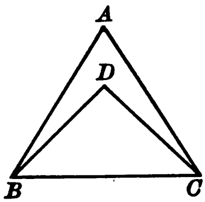

Fig. 79.

7. In Fig. 79, $AB=AC$, and $BD=DC$. Prove that $\angle ABD = \angle ACD$.

8. In Fig. 80, $AC=AB$, and $DC=DB$. Prove that $\angle 1 = \angle 3$.

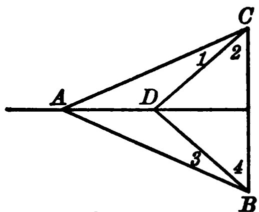

Fig. 80.

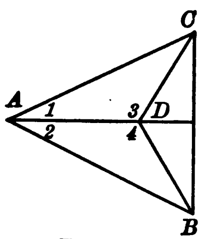

Fig. 81.

9. In Fig. 81, $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$. Prove that $CD=BD$.

10. To cut two converging beams by a line AB which shall make equal angles with them, a carpenter proceeds as follows: He places two squares against the beams (Fig. 82), so that $AO=BO$.

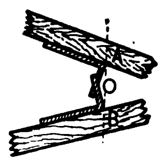
Fig. 82.

Explain why line AB will make equal angles with the two beams.

§ 112. THEOREM IV. ***If two triangles have three sides of one equal respectively to three sides of the other, the two triangles are congruent.***

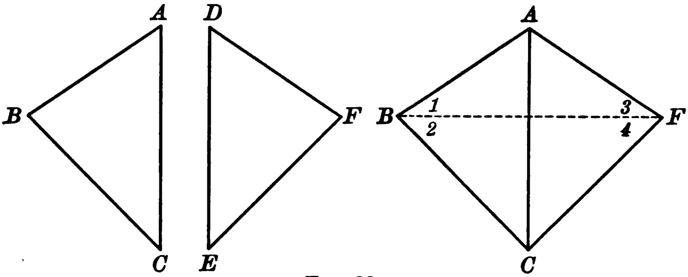

FIG. 83.

In $\triangle ABC$ and DEF, $AC = DE$, $AB = DF$, $BC = EF$.

Imagine $\triangle DEF$ to slide along the page until point D comes on A.

Let $\triangle DEF$ rotate about A until DE coincides with AC.

In the resulting figure at the right, draw BF.

What kind of a $\triangle$ is ABF?

$\angle 1 = \angle 3$ Why?

What kind of a $\triangle$ is CBF?

$\angle 2 = \angle 4$ Why?

$\angle 1 + \angle 2 = \angle 3 + \angle 4$ Why?

$\angle B = \angle F$ Why?

$\triangle ABC \cong \triangle ACF$ Why?

$\triangle ABC \cong \triangle DEF$ Why?

State your conclusion.

NOTE. This theorem is not proved by ***superposition*** for the reason that no angle value is known. If $\angle C$ were known to be equal to $\angle E$, we could overturn $\triangle DEF$ on AC as an axis and then EF would lie along CB. Since $\angle C$ is not known to be equal to $\angle E$, we do not know where EF would lie, hence we do not attempt superposition.

EXERCISES

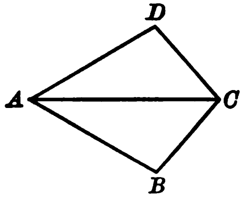

Fig. 84.

1. In Fig. 84, $AB = AD$, and $DC = BC$. Prove that AC bisects the $\angle$s A and C.

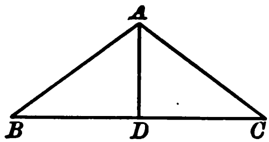

Fig. 85.

2. In Fig. 85, ABC is an isosceles triangle; D is the mid-point of BC; AD is drawn. Prove that AD bisects $\angle A$.

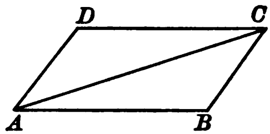

Fig. 86.

3. In Fig. 86, $ABCD$ is a parallelogram in which $AD = BC$, and $AB = CD$. Prove that $\angle B = \angle D$.

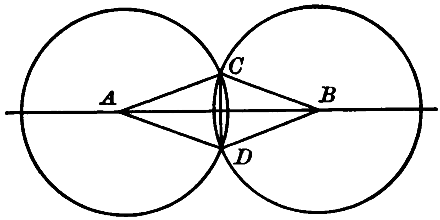

Fig. 87.

4. In Fig. 87, $AC = AD$, and $BC = BD$. Prove that the line through A and B bisects the $\angle$s CAD and CBD.

5. In the $\triangle$s ABC and DEF (Fig. 88), $AB = DE$; $BC = EF$; M and N are mid-points of BC and EF respectively; $AM = DN$. Prove that $\triangle ABM \cong \triangle DEN$.

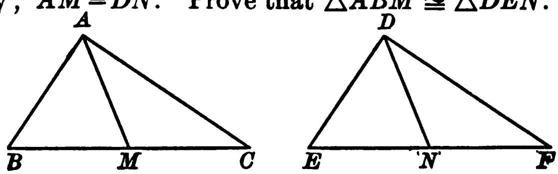

Fig. 88.

6. Fasten three pieces of wood together, by using a nail at each corner, so as to form a triangle (Fig. 89). Explain why this frame is rigid.

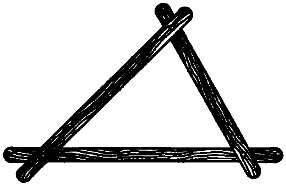

FIG. 89.

7. Why is a roof sufficiently braced when a board is nailed across each pair of rafters?

8. Why are the brace-rods on a bridge always arranged in triangular shapes?

9. Explain why a carpenter can bisect the angle A when he proceeds as follows (Fig. 90): Lay off $AB = AC$. Place a steel square so that $BD = CD$ as shown. Mark D, and then draw AD.

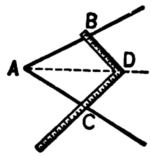

FIG. 90.

10. A simple form of level may be constructed in the following manner: Construct a frame as shown in Fig. 91, with BA equal to BC, and forming the isosceles triangle BDE. Mark Q the mid-point of DE. Suspend a plumb bob from a nail at B. Show that any object upon which the feet A and C rest will be level when the plumb line passes through Q.

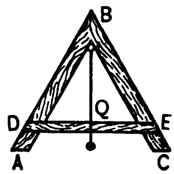

FIG. 91.

11. On page 193, Ex. 1, you constructed with ruler and compasses a given angle at a given point on a given straight line. Prove that $\angle 1$ (Fig. 35) is equal to $\angle MAR$; that is, that the construction is correct.

12. On page 193, Ex. 2, you bisected a given angle. In Fig. 36, prove that $\angle ABX = \angle XBS$.

§ **113**. THEOREM V. *If two angles of a triangle are equal, the sides opposite the equal angles are equal.*

On a piece of tracing paper draw a straight line; on this line mark off with the compasses, a line-segment $XY = 2''$. At X draw an angle of 50°, and at Y an angle of 50°. Complete the $\triangle XYZ$.

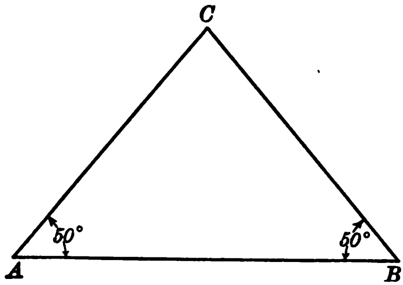

FIG. 92.

Slide $\triangle XYZ$ along the page until point X falls on point B.

Rotate $\triangle XYZ$ about B until XY lies along BA, point Y falling on point A. (Note that the two triangles are on opposite sides of AB.)

Overturn $\triangle XYZ$ on AB as an axis, bringing $\triangle XYZ$ again into the plane of $\triangle ABC$.

Does point Z then fall on point C?

Is $\triangle XYZ \cong \triangle ABC$?

Can $\triangle XYZ$ be made to coincide with $\triangle ABC$ if point X is placed on point A and XY along AB?

Does this prove that $AC = CB$?

State your conclusion.

EXERCISES

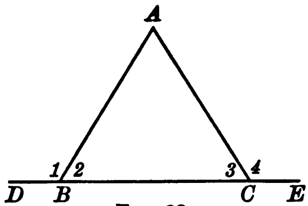

FIG. 93.

1. In Fig. 93, $\angle 1 = \angle 4$.
Prove that $AB = AC$.

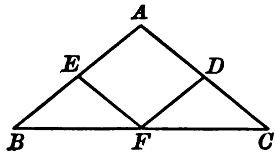

FIG. 94.

2. In Fig. 94, $\angle B = \angle C$.
E is the mid-point of AB.
D is the mid-point of AC.
F is the mid-point of BC.
Prove that $EF = FD$.

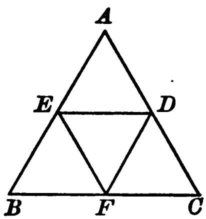

FIG. 95.

3. In Fig. 95, $\triangle ABC$ is equiangular.
E, F, and D are the mid-points of the sides.
Prove that $\triangle EFD$ is equilateral.

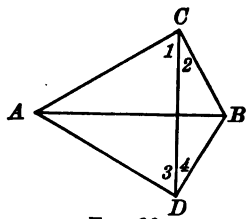

FIG. 96.

4. In Fig. 96, $\angle C = 90°$; $\angle D = 90°$; $\angle 1 = \angle 3$.
Prove that $BC = BD$.
$\angle 2 = \angle 4$. Why?

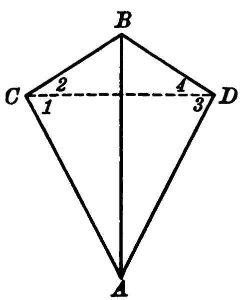

FIG. 97.

5. In Fig. 97, $\measuredangle C$ and D are rt. $\measuredangle$; $AC = AD$.
Prove that $\triangle ADB \cong \triangle ACB$.

NOTE. $ADBC$ is the plan for a *kite*, whose axis is AB.

§ **114.** THEOREM VI. ***If two right triangles have the hypotenuse and a side of one equal respectively to the hypotenuse and a side of the other, the two right triangles are congruent.***

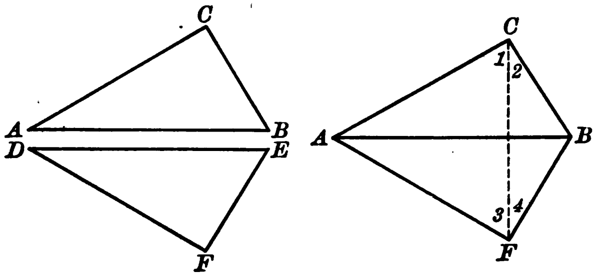

FIG. 98.

In the rt. $\triangle ABC$ and DEF, the hypotenuse $AB =$ hypotenuse DE, and $AC = DF$.

Imagine the rt. $\triangle DEF$ to slide along the page until the point D comes on A.

Let $\triangle DEF$ rotate about A until the hypotenuse DE coincides with the hypotenuse AB. (This gives us the figure at the right.)

In the figure at the right, $\angle C = \angle F$. Why?

Draw CF.

In $\triangle AFC$, $\angle 1 = \angle 3$ ($AF = AC$)

In $\triangle BFC$, $\angle 2 = \angle 4$ (Complements of equal angles)

$BC = BF$ Why?

rt. $\triangle ABC \cong$ rt. $\triangle ABF$ Why?

rt. $\triangle ABC \cong$ rt. $\triangle DEF$ Why?

State your conclusion.

EXERCISES

1. In Fig. 99, OC is $\perp BC$; OA is $\perp BA$; $OC = OA$.

Prove that OB bisects the angle ABC.

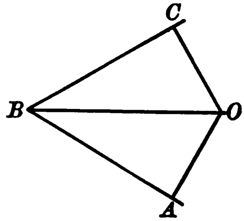

FIG. 99.

2. In the $\triangle ABC$ (Fig. 100), FE and FD are $\perp$ to AB and AC respectively; $FE = FD$; F is the mid-point of BC.

Prove that the $\triangle ABC$ is isosceles.

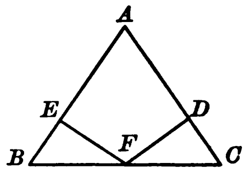

FIG. 100.

3. In Fig. 101, prove that the $\triangle ABC$ is isosceles, if the perpendiculars from the extremities of the base to the opposite sides are equal.

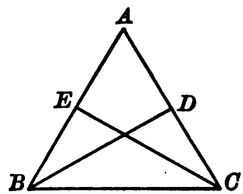

FIG. 101.

4. Prove that every point in the perpendicular bisector of a line is equidistant from the ends of the line.

5. Prove that the altitude upon the base of an isosceles triangle bisects the base; also the vertex angle.

6. To measure the distance from A to B across a river, you may proceed as follows (Fig. 102): Run the line AC at right angles to AB and mark its mid-point O. At C run the line CD at right angles to AC. Locate D in line with points O and B. Then measure CD.

Prove that $CD = AB$.

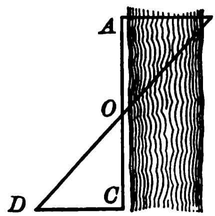

FIG. 102.

CHAPTER XVII

PARALLEL LINES AND PARALLELOGRAMS

§ **115. Parallel Lines.** ***Parallel lines*** are lines that lie in the same plane and do not meet, however far they are extended.

POSTULATE. *Two lines in the same plane perpendicular to the same line are parallel* (||).

CONSTRUCTIONS

1. To construct a line parallel to a given line through a given point.

Carry out this construction with ruler and compasses, as shown in Fig. 103. (See Course II, page 99.)

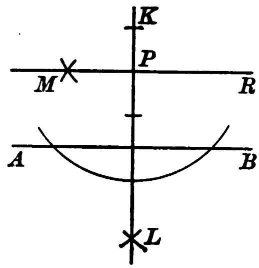

FIG. 103.

2. A draftsman lays off parallel lines by moving his T-square along the straight edge of his drawing board as shown in Fig. 104. Explain.

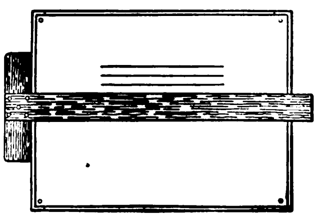

FIG. 104

3. A draftsman lays off vertical parallel lines by using a drawing triangle, the T-square, and drawing board as shown in Fig. 105. Explain.

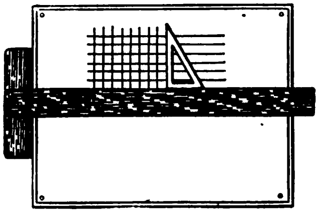

FIG. 105.

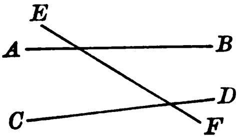

FIG. 106.

§ 116. **Transversal.** A line that crosses two or more lines is called a ***transversal*** of those lines.

In Fig. 106, EF is a transversal of the lines AB and CD.

§ 117. THEOREM I. *If two parallel lines are crossed by a transversal, then all of the acute angles formed are equal, and all of the obtuse angles formed are equal.*

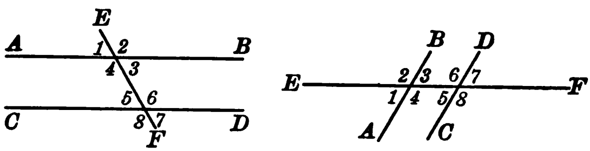

FIG. 107.

Using a protractor, verify in each figure the following equalities:

ACUTE ANGLES	OBTUSE ANGLES
$\angle 1 = \angle 5$	$\angle 2 = \angle 6$
$\angle 7 = \angle 3$	$\angle 4 = \angle 8$
$\angle 1 = \angle 7$	$\angle 2 = \angle 8$
$\angle 5 = \angle 3$	$\angle 4 = \angle 6$

$\measuredangle$ 1 and 5, $\measuredangle$ 4 and 8, $\measuredangle$ 2 and 6, $\measuredangle$ 3 and 7 are called *corresponding* angles.

$\angle$s 4 and 6, $\angle$s 3 and 5 are called *alternate-interior* angles.

$\angle$s 1 and 7, $\angle$s 2 and 8 are called *alternate-exterior* angles.

EXERCISES

1. In Fig. 108, $\angle 1 = 125°$. Find the number of degrees in each of the other angles.

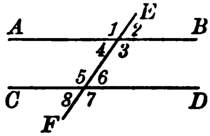

FIG. 108.

2. In Fig. 109, $\angle 6 = 30°$. Find the number of degrees in each of the other angles.

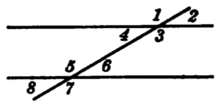

FIG. 109.

3. In Fig. 110, $AB \parallel CD$; $\angle 1 = 25°$; $\angle 3 = 70°$. Find the number of degrees in each of the remaining angles.

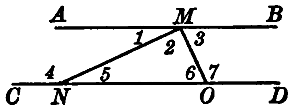

FIG. 110.

4. A line crosses two parallel lines so that one obtuse angle is 30° more than twice one of the acute angles. Find the number of degrees in each angle in the figure.

5. A line crosses two parallel lines so that one of the obtuse angles is 5° more than four times one of the acute angles. Find the number of degrees in each angle in the figure.

6. In Fig. 111, $AB \parallel CD$; $\angle 6 = 100°$; $\angle 7 = 40°$.

How many degrees are there in each of the other angles?

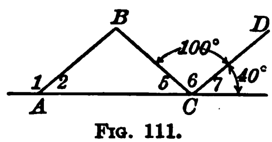

FIG. 111.

7. In Fig. 112, $AB \parallel DE$; $BC \parallel EF$.

Prove that $\angle ABC = \angle DEF$.

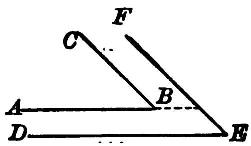

FIG. 112.

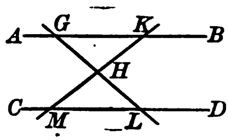

Fig. 113.

8. In Fig. 113, $AB \parallel CD$; $GH = HL$.

Prove that $MH = HK$.

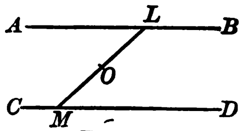

Fig. 114.

9. In Fig. 114, $AB \parallel CD$; O is the mid-point of LM.

Prove that O is also the mid-point of any other transversal through O included between AB and CD.

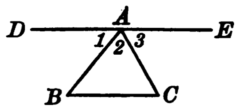

Fig. 115.

10. In Fig. 115, $DE \parallel BC$ through A.

Prove that $\angle A + \angle B + \angle C$ of $\triangle ABC = \angle 1 + \angle 2 + \angle 3$.

§ **118.** Theorem II. ***The sum of the angles of a triangle is equal to* 180°*, or two right angles.***

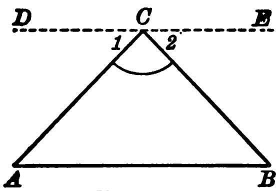

Fig. 116.

Draw the line DE through the vertex C of $\triangle ABC \parallel AB$. Number the angles as in the figure.

$\angle 1 = \angle A$ Why?

$\angle 2 = \angle B$ Why?

$\angle 1 + \angle C + \angle 2 = ?$

$\angle A + \angle C + \angle B = ?$

State the conclusion.

Using the same $\triangle ABC$, can you prove this theorem by drawing a line through the vertex $B \parallel AC$? Try it.

EXERCISES

1. In Fig. 117, $\angle C = 90°$; $\angle A = 35°$; $\angle B = ?$ The two acute angles of a right triangle are complementary. Why?

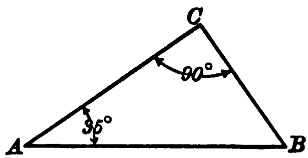

FIG. 117.

2. Prove that if two angles of one triangle are equal to two angles of a second triangle, the third angles are also equal.

3. If two right triangles have the hypotenuse and an acute angle of the one equal respectively to the hypotenuse and an acute angle of the other, the two triangles are congruent. (Apply Ex. 2.)

4. In a certain right triangle one acute angle is twice as large as the other. How many degrees are there in each acute angle?

5. In a certain right triangle one acute angle is three times as large as the other. How many degrees are there in each acute angle?

6. In a certain isosceles triangle the vertex angle is twice as large as a base angle. Find the number of degrees in each angle of the isosceles triangle.

7. Find the value of each angle of an equilateral triangle.

8. Prove that an exterior angle of a triangle is equal to the sum of the two opposite interior angles.

(SUGGESTION. In Fig. 116, extend AB through B; draw a line through $B \parallel AC$.)

9. ABC (Fig. 118) is an equilateral $\triangle$; $CD \perp AB$.

Find the values of $\angle$ 1 and 2.

Prove that $AD = DB$.

State your conclusion.

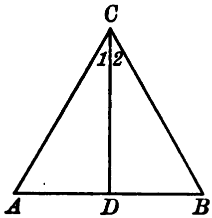

FIG. 118.

10. One angle of a triangle is 40°. The other two angles have the ratio 4 to 1. Find the other two angles, and draw a figure to represent such a triangle. (The equation is: $A + 4A + 40 = 180$.)

11. One angle of a triangle is 50°. The other two angles have the ratio 1.5. Find the other two angles. (The equation is: $A + 1.5A + 50 = 180$.)

12. In an isosceles triangle, each of the angles at the base is 10° more than twice the third angle.

(*a*) Find the three angles. (Represent the angles by A, $2A + 10$, and $2A + 10$. Why?)

(*b*) Draw the figure, making the base 2.6″.

13. In a right triangle one acute angle is twice the other; the shortest side is 1″.

(*a*) Find the two acute angles.

(*b*) Draw the triangle, using the shortest side for the base, with the right angle at its left, as in $\triangle ABC$, Fig. 119.

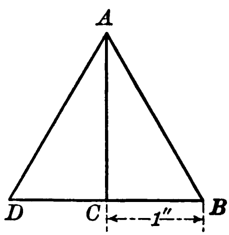

FIG. 119.

(*c*) Draw another right triangle with the same parts and place it in the position of $\triangle ADC$, Fig. 119.

(*d*) Test the sides and angles of $\triangle ABD$. Name the triangle.

(*e*) How does AB compare with BC?

It follows that: *In a right triangle if one acute angle is twice the other, then the hypotenuse is twice the shortest side.* This is the 60°–30° right triangle used by the draftsman. (See Fig. 121, page 228.)

§ **119**. Theorem III. *If two lines are crossed by a transversal so that either of the following pairs of angles are equal, the two lines are parallel:*

(*a*) *A pair of alternate-interior angles; or*

(*b*) *A pair of corresponding angles.*

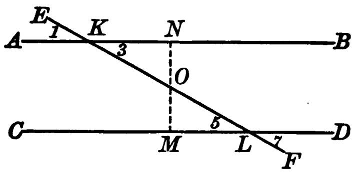

Fig. 120.

The transversal EF crosses the two lines AB and CD so that,

(*a*) a pair of alternate-interior angles are equal; that is, $\angle 3 = \angle 5$.

O is the mid-point of KL.

NM is drawn through $O \perp CD$.

$\triangle OML \cong \triangle ONK$	Why?
$\angle OML =$ a rt. $\angle$	Why?
$\angle KNO = \angle OML$	Why?
$AB \parallel CD$	(Postulate, page 220.)

(*b*) a pair of corresponding angles are equal; that is, $\angle 1 = \angle 5$.

$\angle 1 = \angle 3$	Why?
$AB \parallel CD$	Why?

State your conclusion.

EXERCISES

1. Prove that if a line is perpendicular to one of two parallel lines, it is perpendicular to the other one also.

2. Figure 121 shows a draftsman's triangle which is used for drawing perpendicular lines and parallel lines.

If the hypotenuse is 8″, how long is the shortest side? How long is the third side?

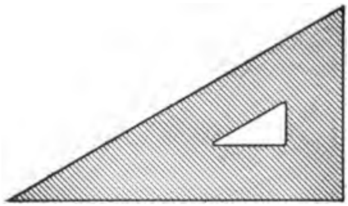

FIG. 121.

3. In order to draw a line parallel to a line l (Fig. 122) through a point P, a draftsman places a drawing triangle

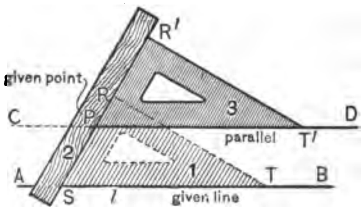

FIG. 122.

so that one side coincides with l, and the other side passes through P. He then lays a ruler against the side of the triangle that passes through P, and finally slides the triangle along the edge of the ruler, until one vertex of the triangle comes to P. Why will a line drawn along the side of the triangle, originally in coincidence with l, be the parallel to l through P?

§ 120. Classification of Quadrilaterals. A ***quadrilateral*** is a plane figure inclosed by four straight lines.

Quadrilaterals are divided into three groups:

Parallelograms, having opposite sides parallel.

Trapezoids, having only two sides parallel.

Trapeziums, having no two sides parallel.

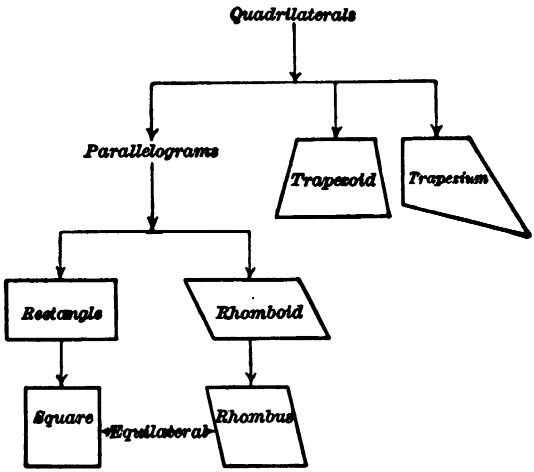

FIG. 123.

Parallelograms are divided into two groups:

(*a*) ***Rectangles***, having all angles right angles.
A ***square*** is an equilateral rectangle.

(*b*) ***Rhomboids***, having no angles right angles.
A ***rhombus*** is an equilateral rhomboid.

§ 121. ***Parallelograms.*** A ***parallelogram*** is a quadrilateral whose opposite sides are parallel.

§ **122.** THEOREM IV. ***In any parallelogram, either diagonal divides it into two congruent triangles, and the opposite sides of the parallelogram are equal.***

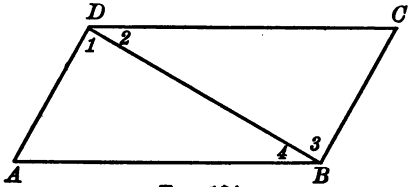

FIG. 124.

In ▱ $ABCD$ draw diagonal DB.

Compare the $\triangle ABD$ with $\triangle BCD$.

$\angle 4$ of $\triangle ABD = \angle 2$ of $\triangle BDC$ Why?

$\angle 1$ of $\triangle ABD = \angle 3$ of $\triangle BDC$ Why?

$BD \equiv BD$

$\triangle ABD \cong \triangle DBC$ Why?

What follows as to AB and CD? AD and BC?

EXERCISES

1. Prove that the opposite angles of a parallelogram are equal.

2. Prove that parallel lines which are included between parallel lines are equal.

3. Prove that the sum of any two consecutive angles of a parallelogram is equal to 180° (Fig. 125).

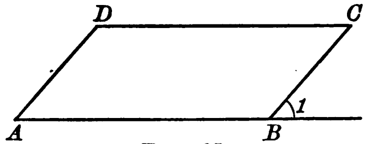

FIG. 125.

(SUGGESTION. $\angle 1 = \angle C$. Why?)

4. Prove that if one angle of a parallelogram is a right angle all the angles are right angles.

5. Prove that if two adjacent sides of a parallelogram are equal, all of its sides are equal.

§ **123**. Theorem V. *The diagonals of a parallelogram bisect each other.*

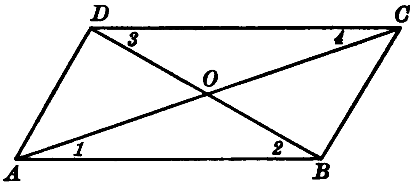

Fig. 126.

In ▱ $ABCD$ draw the diagonals AC and BD.

Call their point of intersection O.

Compare $\triangle ABO$ with $\triangle CDO$.

$AB = DC$	Why?
$\angle 1 = \angle 4$	Why?
$\angle 2 = \angle 3$	Why?
$\triangle ABO \cong \triangle CDO$	Why?
$BO = OD$	Why?
$AO = OC$	Why?

State your conclusion.

Could you have used $\triangle ADO$ and BCO? Try them.

EXERCISES

1. Prove that the diagonals of a rectangle are equal.

2. Prove that the diagonals of a square bisect each other at right angles.

(Suggestion. Each diagonal is the hypotenuse of an isosceles right triangle.)

3. In Fig. 127, XY, DE, and BC are parallel lines.

D is the mid-point of AB.

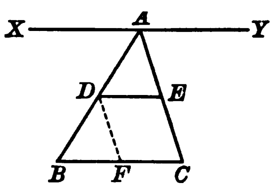

Fig. 127.

Prove that E is the mid-point of AC.

Draw $DF \parallel AC$. Then prove,

(*a*) that $\triangle ADE \cong \triangle BFD$,

(*b*) that $DF = AE$,

(*c*) that $EC = DF$,

(*d*) that $AE = EC$. Why?

4. Using Fig. 127, prove that $DE = \frac{1}{2}BC$.

§ **124**. THEOREM VI. *If a quadrilateral has each pair of its opposite sides equal, it is a parallelogram.*

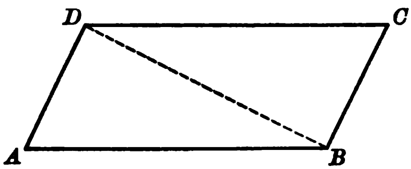

FIG. 128.

$ABCD$ is a quadrilateral in which $AB = DC$, and $AD = BC$.

Draw the diagonal BD.

$\triangle ABD \cong \triangle BCD$	Why?
$\angle ABD = \angle BDC$	Why?
$AB \parallel CD$	Why?
$\angle ADB = \angle DBC$	Why?
$AD \parallel BC$	Why?
$ABCD$ is a $\square$.	Why?

State your conclusion.

EXERCISE

If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

§ **125. Polygons.** A ***polygon*** is a plane figure inclosed by any number of straight lines. A polygon of five sides is a *pentagon;* of six sides, a *hexagon;* of seven sides, a *heptagon;* etc.

§ **126.** Theorem VII. *The sum of all the angles of a polygon is equal to* $(n-2)180°$, *when n equals the number of sides of the polygon.*

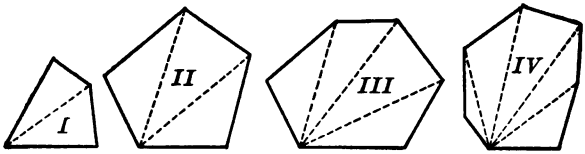

Fig. 129.

In the quadrilateral (I) there are two triangles, or $(n-2)$ ▲.

In the pentagon (II) there are three triangles, or $(n-2)$ ▲.

In the hexagon (III) there are four triangles, or $(n-2)$ ▲.

In the heptagon (IV) there are five triangles, or $(n-2)$ ▲, etc.

The sum of the angles of all of the triangles in each polygon is equal to the sum of the angles of that polygon.

What is the sum of the angles of a triangle? Then what follows? State your conclusion.

EXERCISES

1. Express the sum of all the angles of a polygon of 7 sides; of 9 sides; of 10 sides; of 12 sides.

2. How many degrees are there in each angle of an equiangular quadrilateral? Pentagon? Hexagon? Octagon? Decagon?

CHAPTER XVIII

CIRCLES

§ **127. Definitions.** A ***circle*** is a curve all points of which are equally distant from a point within called the ***center***. This curve is often called the ***circumference*** of the circle.

An ***arc*** of a circle is any portion of its circumference (Fig. 130).

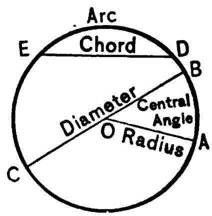

FIG. 130.

The ***radius*** of a circle is a straight line drawn from the center to any point in its circumference.

A ***diameter*** is a straight line drawn through the center of the circle and terminating in the circumference.

A ***chord*** is a straight line joining any two points of the circumference.

A ***semicircle*** is one half of a circle.

A ***central angle*** is the angle between any two radii.

§ **128. Central Angles.** In Fig. 131, the central angle AOB is said to ***intercept*** (*cut off*) the arc AB; while the arc AB is said to ***subtend*** (*stretch across*) the angle AOB.

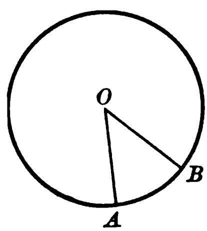

FIG. 131.

A circle is generated when a segment of a line turns in a plane about one end as a point through a complete revolution. The circumference is described by the other end of the line. Any two positions of the line-seg-

ment, or radius, form a central angle. Hence it follows that:

(1) *In the same circle (or equal circles) if two central angles are equal, they intercept equal arcs,* for the radius, while turning through equal angles, describes equal arcs of the circumference.

(2) *In the same circle (or equal circles) if two arcs are equal, they subtend equal central angles,* for the radius, while describing equal arcs of the circumference, turns through equal angles.

§ 129. THEOREM I. ***If the diameter of a circle is perpendicular to a chord, it bisects that chord.***

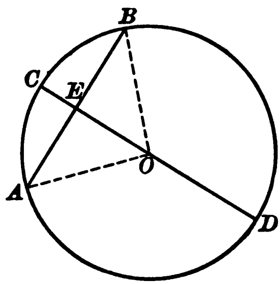

FIG. 132.

In the circle O, the diameter CD is $\perp$ to the chord AB at E.

Draw the radii OA and OB.

Prove that rt. $\triangle AOE \cong$ rt. $\triangle EOB$.

$$AE = EB \qquad \text{Why?}$$

State your conclusion.

EXERCISES

1. Draw a circle having a 10″ radius to the scale of 10 to 1. In the circle draw a chord 16″ long, and draw a line from the center of the circle perpendicular to the chord. Find the distance from the center to the chord.

(SUGGESTION. Draw a radius to the end of the chord. The radius is the hypotenuse of a right triangle.)

2. In a circle whose radius is 12 cm., a chord 9 cm. long is drawn. How far is the chord from the center?

3. In a 16″ circle (diameter = 16″) a chord is at a distance of 3″ from the center. How long is the chord?

4. In a 24 cm. circle a chord is 9 cm. from the center. How long is the chord?

5. In Fig. 133, the chords AB and AC make equal angles with the radius OA. OM and ON are $\perp$ to AB and AC, respectively.

Prove, $AM = AN$; also, $AB = AC$.

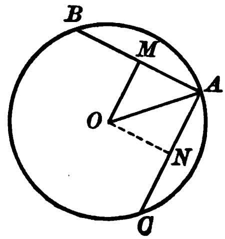

FIG. 133.

§ **130. Tangents.** A ***tangent*** to a circle is a straight line of unlimited length that touches the circle at only one point. This point is called the *point of contact.*

In Fig. 134, AB is *tangent* to the circle O, at A. A is the *point of contact.*

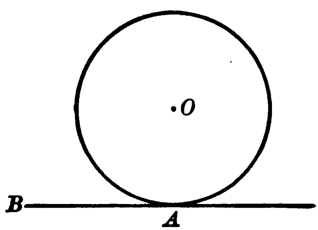

FIG. 134.

§ **131.** Theorem II. ***A line perpendicular to a radius of a circle at its extremity is tangent to the circle at that point.***

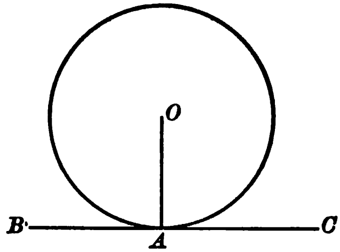

Fig. 135.

Is OA the shortest line from O to BC? Why?

Are all points on the circle at the distance OA from O? Why?

Can any point other than A in the line BC, be on the circle?

State your conclusion.

EXERCISES

1. The radius of a circle is 8″ (Fig. 136). The tangent to the circle from A is 6″. Find the distance (OA) from the center to the end of the tangent.

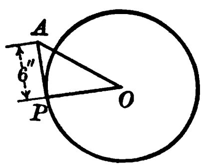

Fig. 136.

2. The distance from the center of a circle to a point outside is 10 cm. and the radius of the circle is 5 cm. Find the length of the tangent from that point.

3. A point is 8″ from the center of a 12″ circle. Find the length of the tangent from that point.

4. At a point on a circle construct a tangent to the circle.

(SUGGESTION. Draw the radius to the given point.)

5. Prove that two tangents drawn to a circle at the extremities of a diameter are parallel. (See Postulate, page 221.)

6. If two tangents are drawn to a circle from an external point, prove that they are equal (Fig. 137). Draw

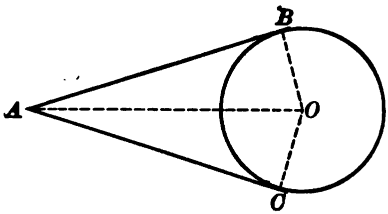

FIG. 137.

AO, *CO*, and *BO*. What kind of triangles are formed? Are these triangles congruent?

7. In Fig. 137, Ex. 6, prove that $\angle BAO = \angle OAC$.

8. To locate the center of circular objects the machinist uses an instrument called a *center square* (Fig. 138).

The center square consists of two arms *AK* and *AN* forming an angle *NAK*. Through the vertex of this angle there passes a steel blade which bisects the angle. If the arms are adjusted to touch the circular object, will this blade pass through the center of the object? Why?

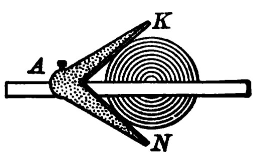

FIG. 138.

§ 132. **Measurement.** The ***numerical measure*** of a quantity is the *number of times* that it contains a given ***unit of measure.***

For example, (*a*) A line is 5 feet long. The unit of measure is the *foot*, and the numerical measure is the number 5.

(*b*) The area of a floor is 24 square yards. The unit of measure is the *square yard*, and the numerical measure is the number 24.

(*c*) A revolution contains 360 degrees. The unit of measure is the *angle of one degree*, and the numerical measure is the number 360.

(*d*) A circle contains 360 degrees. The unit of measure is the *arc of one degree*, and the numerical measure is the number 360.

With the units of measure named in (*c*) and (*d*), *a revolution has the same numerical measure as the circle.*

In generating a circle, as the radius turns through a given angle, the arc described is the same fractional part of the circle that the given angle is of a revolution; hence

A central angle has the same numerical measure as its intercepted arc. That is, a central angle of 45° intercepts an arc of 45°.

§ 133. **Inscribed angles.** An ***inscribed angle*** is an angle whose vertex is on the circle and whose sides are chords.

An angle is ***inscribed in an arc*** when its vertex is on the arc and its sides pass through the extremities of the arc.

In Fig. 139, $\angle 1$ is an *inscribed angle.* It is *inscribed in the arc* *ABC.*

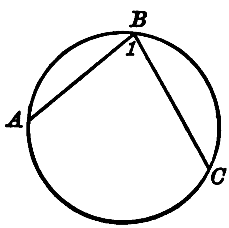

FIG. 139.

§ **134.** THEOREM III. ***An inscribed angle has the same numerical measure as one half its intercepted arc.***

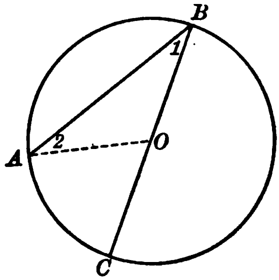

FIG. 140.

CASE I. In Fig. 140, $\angle ABC$ is an inscribed angle, having the diameter BC for one of its sides. Draw the radius AO.

$\angle AOC = \angle 1 + \angle 2$ (*An exterior angle of a triangle is equal to the sum of the two opposite interior angles.* Ex. 8, page 225.)

$\angle 1 = \angle 2$ Why?

$\angle AOC = \angle 1 + \angle 1$ (Substitution)

$\frac{1}{2} \angle AOC = \angle 1$ Why?

$\angle AOC$ has the same numerical measure as arc AC. Why?

$\angle 1$ has the same numerical measure as $\frac{1}{2}$ the arc AC. Why?

But, since $\angle ABC \equiv \angle 1$, $\angle ABC$ has the same numerical measure as $\frac{1}{2}$ the arc AC.

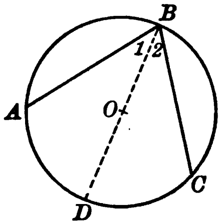

FIG. 141.

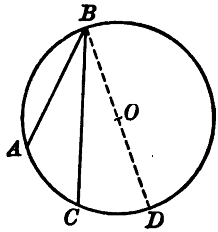

FIG. 142.

CASE II. In Fig. 141, the center of the circle is within the sides of the inscribed angle ABC. Draw BD, the diameter through B.

$\angle 1$ has the same numerical measure as $\frac{1}{2}$ the arc AD. (Case I)

$\angle 2$ has the same numerical measure as $\frac{1}{2}$ the arc DC. Why?

Adding the numerical measures of the angles and of the arcs, $\angle ABC$ has the same numerical measure as $\frac{1}{2}$ the arc AC.

CASE III. In Fig. 142, the center of the circle is without the sides of the inscribed angle ABC. Draw BD, the diameter through B.

$\angle ABD$ has the same numerical measure as $\frac{1}{2}$ the arc ACD. Why?

$\angle CBD$ has the same numerical measure as $\frac{1}{2}$ the arc CD. Why?

Subtracting the numerical measures of the angles and of the arcs, $\angle ABC$ has what numerical measure?

State the conclusion about the measure of any inscribed angle.

EXERCISES

1. An angle inscribed in a semicircle is a right angle. In Fig. 143, how many degrees are there in arc ADC? $\angle ABC = 90°$. Why?

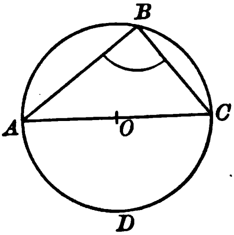

FIG. 143.

2. In Fig. 144, P is a point outside the circle whose center is O. PK is constructed tangent to the circle from P.

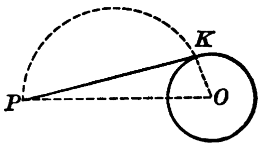

FIG. 144.

NOTE. PO is the diameter of the circle used in the construction of the tangent PK.

Prove that PK is tangent to the circle at K.

3. Pattern makers use a carpenter's square to determine a semicircle. The square is placed as in Fig. 145. If the heel of the square touches the bottom of the hole for all positions of the square, while the sides rest against the edges of the hole, then the hole is a semicircle. Why?

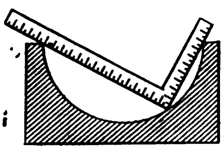

FIG. 145.

4. Show how to locate the diameter of a given circle by applying a rectangular sheet of paper to the circle (Fig. 146). How would you locate the center of the circle?

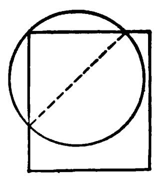

FIG. 146.

5. A surveyor desires to lay out a line at 90° to the line AB (Fig. 147). He sets a stake at a convenient point P, 50 feet from a stake at B. With one end of the 50-foot tape at P, he describes an arc with the other end, thus locating a stake at C. Keeping one end still at P, with the other end he locates a stake at D, so that C, P, and D are in the same straight line. Why is BD perpendicular to BC?

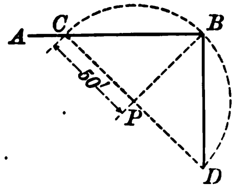

FIG. 147.

CHAPTER XIX

SIMILAR TRIANGLES

§ 135. Similar Triangles. ***Similar triangles*** are triangles whose corresponding angles are equal and whose corresponding sides are in proportion.

§ 136. THEOREM I. ***If two triangles have their corresponding angles equal, they are similar.***

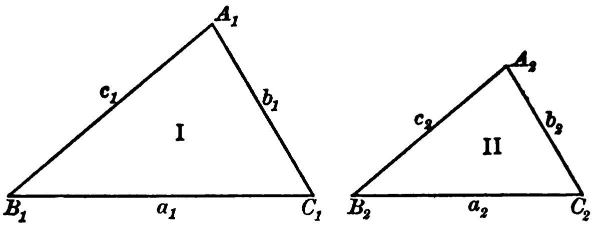

FIG. 148.

Draw a triangle (I) having angles of 40°, 60°, and 80°, making the longest side 1.6″ long. Draw another triangle (II) having angles of 40°, 60°, and 80°, making the longest side 1.2″ long.

Letter the triangles as in Fig. 148.

Measure a_1 and a_2 and find the ratio of a_1 to a_2.

In a similar way find the ratio of b_1 to b_2, and c_1 to c_2.

How do these ratios compare in value?

Are the triangles similar?

In similar triangles, the pairs of corresponding sides are opposite the pairs of equal angles.

EXERCISES

1. Two triangles have two angles of one equal to two angles of the other. Are the third angles equal? Are the triangles similar?

2. Two right triangles have an acute angle of one equal to an acute angle of the other. Are the remaining acute angles equal? Are the right triangles similar?

§ 137. Summary.

(1) Two triangles are similar when the three angles of one equal the three angles of the other.

(2) Two triangles are similar when two angles of one equal two angles of the other.

(3) Two right triangles are similar when an acute angle of one equals an acute angle of the other.

EXERCISES

1. The triangles in Fig. 149 are similar; hence $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

In Fig. 149, $x=1.8''$, $y=2.4''$, $z=1.5''$, and $a=1.2''$. (Scale 2 to 1.) Find b and c.

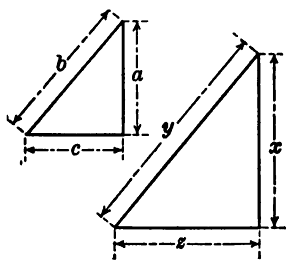

FIG. 149.

The equations are:

$$\frac{1.8}{1.2}=\frac{2.4}{b}, \text{ and } \frac{1.8}{1.2}=\frac{1.5}{c}.$$

Test your answers by measuring b and c.

Using Fig. 149 for reference, copy the following table and fill in the required values. (Estimate your results first.)

	x	y	z	a	b	c
2.	8″	9″	11″	16″		
3.	3.5″	5″	7.5″		15″	
4.	18 cm.			3 cm.	4 cm.	5 cm.
5.		6″		17″	18″	4″
6.	5′	6′	7′	8′		
7.	9 cm.	12 cm.	13 cm.			18 cm.
8.	5.2 cm.	3.5 cm.	6.4 cm.		8.7 cm.	

9. A vertical rod 6′ long casts a shadow 8′ long. How high is a tree that casts a 40′ shadow? (See Fig. 150.)

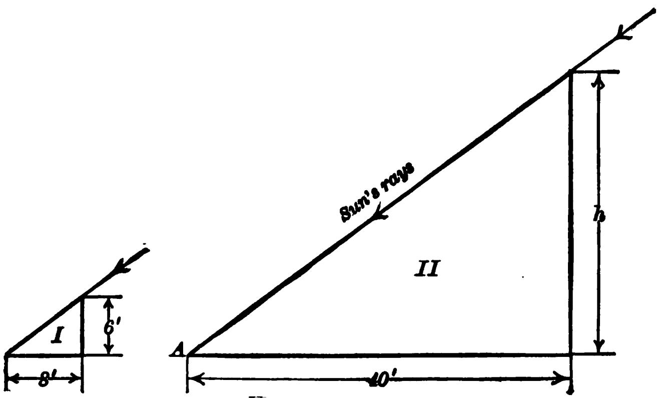

Fig. 150

A ray of sunlight from the top of each object locates the end of its shadow and right triangles are formed. Why are the triangles right triangles?

At the same hour the angles formed by the sun's rays must be equal. What do you know about the right triangles? What do you know about the corresponding sides of these triangles? State an equation for finding h and solve it.

NOTE. The angle A is called the *angle of elevation of the sun.*

10. The shadow cast by a 4′ vertical rod is $5\frac{1}{2}'$, when the shadow cast by a church spire is 220′. How high is the spire?

11. Estimate the height of your school building. By the method in Ex. 9, find the height to the nearest foot. Find the per cent error in your estimate.

12. A monument casts a shadow 118′ long, when a yardstick casts a 40″ shadow. Find the height of the monument to the nearest foot.

13. Estimate the heights of poles and buildings and test your estimates by the method in Ex. 9. Bring into class at least two such problems worked out.

14. The height of a certain hill is desired. At a place A (Fig. 151), the angle of elevation of the top of the hill is 30°. At a place B along a level road, 300 feet nearer the base of the hill, the angle of elevation of the top is 45°. Draw a plan to the scale of 200 feet to one inch. Find, from the plan, the number of feet in the height of the hill.

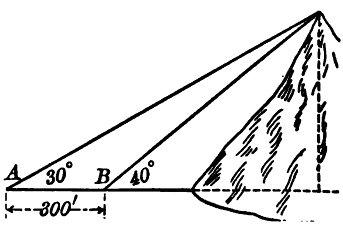

FIG. 151.

15. The distances between two forts, F_1 and F_2, within the enemy's lines, also their distances from the two points of observation, A and B, are desired. Certain angles are measured and found to be as indicated in Fig. 152. The distance AB is 1500 feet. Draw a plan to the scale of 400 feet to one inch. Find from the plan the distances F_1F_2, AF_1, AF_2, BF_1, BF_2.

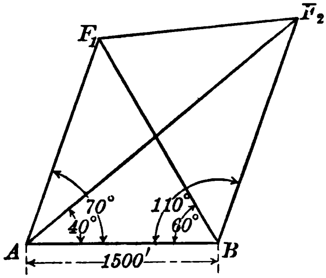

FIG. 152.

§ **138. Tangent Ratio.** On pages 246–247 we found the heights of objects by measuring their shadows. A more practical method will now be shown.

It is as follows: measure the angle of elevation of the top of the object and the horizontal distance from the point of observation to the base of the object; read the tangent ratio of the angle of elevation from a table of tangents (see page 291); compute the height of the object by substituting the values of *tan* A and b in the formula given below.

In Fig. 153, a is the segment of a tangent, to a circle whose radius is b, which is cut off by the central angle A. The ratio of this segment to the radius, $\frac{a}{b}$, is called the ***tangent ratio*** of the angle A.

The formula is written:

$$\tan A = \frac{a}{b}$$

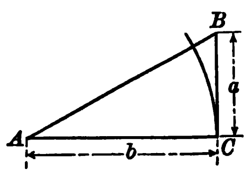

FIG. 153.

For measuring angles the engineer uses an instrument called a *transit* (Fig. 154). With this instrument angles in the vertical plane can be measured on the vertical circle by turning the telescope up and down; angles in the horizontal plane can be measured on the horizontal plate by turning the upper part of the transit to the right and left.

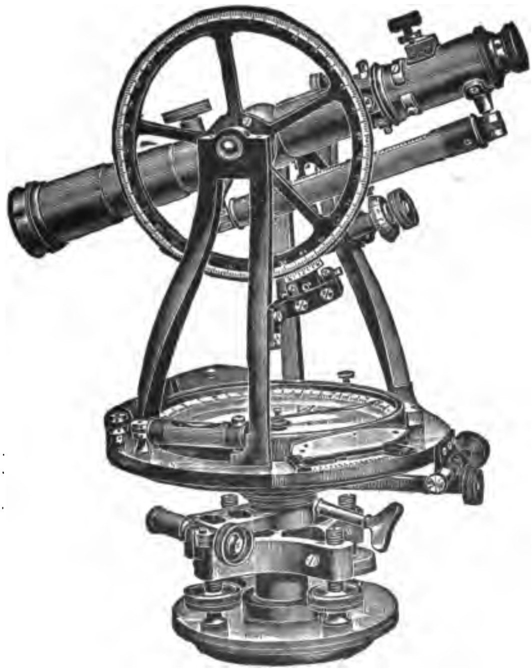

FIG. 154.

EXERCISES

1. In Fig. 155, three right triangles are drawn, all having the same acute angle. $\angle A = 40°$.

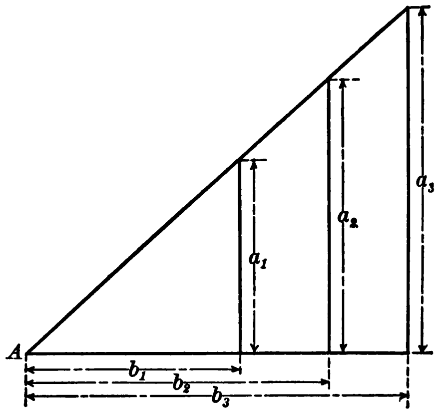

FIG. 155.

Copy the following table. In Fig. 155, measure the lines designated in the table, to the nearest hundredth of an inch. Fill in the remaining values, and find the tangent ratios to two decimal places.

$a_1 = ?$ $a_2 = ?$ $a_3 = ?$	$b_1 = ?$ $b_2 = ?$ $b_3 = ?$	$\frac{a_1}{b_1} = ?$ $\frac{a_2}{b_2} = ?$ $\frac{a_3}{b_3} = ?$
		Average = ?

2. Draw a right triangle having $\angle A = 40°$ and mark a for the side opposite $\angle A$, and b for the other side. Measure a and b to the nearest hundredth of an inch. Find

the ratio of a to b and express it as a decimal to two places. Test your ratio and the average obtained in Ex. 1 for equality.

However large the right triangle is, if the $\angle A = 40°$, the ratio $\frac{a}{b}$ will always have the same value.

On page 291, a table of tangents of various angles is given to three decimal places for angles from 1° to 89°.

Compare your values of tangent 40° (Exs. 1 and 2) with the value given in the table.

NOTE. The study and use of this ratio and other similar ratios is called ***Trigonometry***. The table on page 291 also gives the values of two of the other ratios, *sines* and *cosines*.

§ 139. Table of Tangents. The ***angle of elevation*** (or ***depression***) of an object is the angle that a line from the point of observation to the object makes with the horizontal line in the same vertical plane. It is called the *angle of elevation* if the object is above the eye of the observer; it is called the *angle of depression* if the object is below the eye of the observer.

In Fig. 156, if the observer is at A and the object is at B, $\angle 1$ is the *angle of elevation*. If the observer is at B and the object is at A, $\angle 2$ is the *angle of depression*. Note that $\angle 2 = \angle 1$.

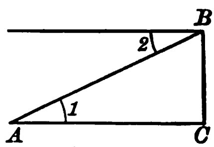

FIG. 156.

EXAMPLE 1. A tree casts a shadow 45 ft. long on a horizontal plain, when the sun is 38° above the horizon. Find the height of the tree.

(SUGGESTION. The angle of elevation of the sun is 38°.)

The solution with diagram (Fig. 157) follows on page 252.

SOLUTION. Make b represent 45′, using a scale of 20′ to 1″. Construct $\angle A = 38°$; erect $a \perp$ to b at the end of b (Fig. 157). Measure a with a ruler; change the measurement to feet and record it for your estimate. Substitute 38 for A and 45 for b in the formula, $\tan A = \frac{a}{b}$.

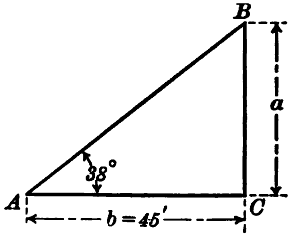

FIG. 157.

①	$\tan 38 = \frac{a}{45}$	(*Est.* $a = 35'$)
②	$0.781 = \frac{a}{45}$	① ≡ (Table)
③	$35.1 = a$	② × 45

CHECK. Compare 35.1 with the measured length of a.

Ans. 35.1′.

EXAMPLE 2. A 9-foot pole casts a shadow 7.6′ long. Find the angle of elevation of the sun within one degree.

SOLUTION. Make b represent 7.6′; draw $a \perp$ to b at the end of b, making it represent 9′ (Fig. 158). Measure $\angle A$ with a protractor and record the result. Substitute 9 for a and 7.6 for b in the formula, $\tan A = \frac{a}{b}$.

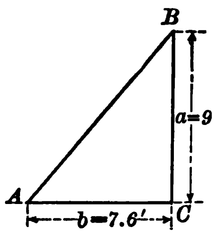

FIG. 158.

①	$\tan A = \frac{9}{7.6}$	(*Est.* $A = 50°$).
②	$\tan A = 1.184$	① ≡
③	$A = 50$	② ≡ (Table)

CHECK. Compare 50 with the measured value of $\angle A$.

Ans. 50°.

EXERCISES

For each of the following exercises, first draw a diagram to a convenient scale; then measure the required part for your estimate.

1. At a horizontal distance 423′ from the base of a tower the angle of elevation of the top is 57°. Find the height of the tower.

2. A vertical pole $24\frac{1}{2}'$ high casts a horizontal shadow $43\frac{1}{2}'$ long. Find the angle of elevation of the sun.

NOTE. In exercises where the angle is required, find the angle to within one degree.

3. At a horizontal distance of 137′ from the foot of a steeple the angle of elevation of its top is 60°. Find the height of the steeple.

4. From the top of a cliff that rises 215′ out of the water the angle of depression of a boat is 23°. Find the distance of the boat from the foot of the cliff.

5. A tower 37.5 meters high is situated on the bank of a river. The angle of depression of an object on the opposite bank is 32°. Find the width of the river.

6. The angle of elevation of an airplane at a point A on level ground is 64°. The point M on the ground directly under the airplane is 225 yd. from A. How high is the airplane?

7. The beam of a searchlight on a tower shines directly on a boat. The angle of depression is 37° and the searchlight is 420 meters above the horizontal. How far is the boat from the tower?

8. From the top of a cliff 145′ high, on one side of a river, the angle of depression of an object on the bank opposite is 24°. How wide is the river?

9. The angle of elevation of the sun is 42°. How tall is a tree that casts a shadow on level ground 72′ long?

10. A distance MN along the bank of a river (Fig. 159) is found by measurement to be 127′. A tree at O on the other bank is directly opposite M, so that OM is at right angles to MN. The angle MNO is 37°. Find the width of the river at M.

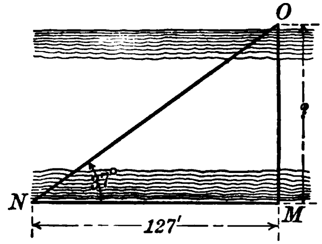

FIG. 159.

11. Find the angle of elevation of an inclined plane if it rises 18′ in a horizontal distance of 40′.

12. A railroad incline rises 20′ in every 100′ along the horizontal. Find the angle of elevation of the road.

13. The grade of a railroad is $2\frac{1}{2}$ per cent; that is, it rises $2\frac{1}{2}'$ for every horizontal distance of 100′. Find the angle of elevation.

14. Find the angle of elevation of a 3.5 per cent grade.

15. A railway grade is 200′ to the mile. What is the angle of elevation of the road bed?

16. Find the angle between the rafter and the span in a $\frac{1}{2}$ pitch roof. (The *pitch* is the ratio of the "rise" to the "span." See Fig. 160.)

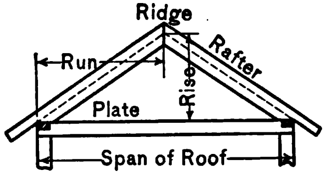

FIG. 160.

17. Find the angle between the rafter and the span in a $\frac{1}{3}$ pitch roof.

§ 140. THEOREM II. *In any right triangle, the altitude upon the hypotenuse divides the triangle into two right triangles which are similar to the given triangle and to each other.*

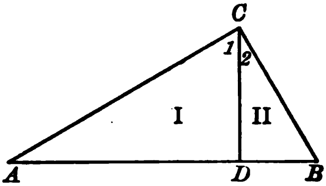

FIG. 161.

In the rt. $\triangle ABC$, CD is the altitude upon the hypotenuse AB.

Comparison of triangles:

(1) $\triangle$I with $\triangle ABC$.

$\angle A \equiv \angle A$

$\angle 1 = \angle B$ Why?

$\triangle \text{I} \sim \triangle ABC$ Why?

(2) $\triangle$II with $\triangle ABC$.

$\angle B \equiv \angle B$

$\angle 2 = \angle A$ Why?

$\triangle \text{II} \sim \triangle ABC$ Why?

(3) $\triangle$I with $\triangle$II.

$\angle A = \angle 2$

$\angle 1 = \angle B$

$\triangle \text{I} \sim \triangle \text{II}$ Why?

§ **141**. THEOREM III. *In any right triangle, the altitude upon the hypotenuse is a mean proportional between the segments of the hypotenuse; also, either side of the right triangle is a mean proportional between the hypotenuse and the segment adjacent to that side.*

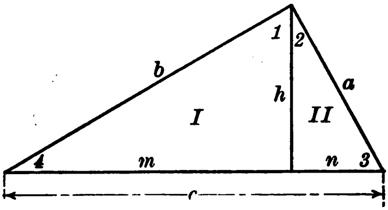

FIG. 162.

Parts of $\triangle$I compared with parts of $\triangle$II.

$\angle 1 = \angle 3$ (Both complements of $\angle 2$)

$\angle 4 = \angle 2$ (Both complements of $\angle 1$)

m corresponds to h.

h corresponds to n.

Then, $$\frac{m}{h} = \frac{h}{n}$$ Why?

State your conclusion.

Parts of $\triangle$I compared with parts of given triangle.

$$\angle 4 \equiv \angle 4$$

$$\angle 1 = \angle 3$$

Then, $$\frac{m}{b} = \frac{b}{c}$$

State your conclusion.

EXERCISES

1. Prove that a (Fig. 162) is a mean proportional between the hypotenuse and the segment adjacent to a.

2. In a right triangle whose hypotenuse is 25″, one segment of the hypotenuse made by the altitude upon it is 5″. Find the altitude.

3. The hypotenuse of a right triangle is 13 cm. and the altitude upon the hypotenuse is 6 cm.

(a) Find the segments of the hypotenuse.

(b) Find each side of the given right triangle.

4. The hypotenuse of a right triangle is 18.4″, and the altitude upon the hypotenuse is 7.2″.

(a) Find the segments of the hypotenuse.

(b) Find each side of the given right triangle.

5. The diameter of a circle is 40″, and the perpendicular from a point in the circumference upon the diameter is 12″. Find the segments of the diameter formed by the perpendicular (Fig. 163).

(Suggestions. $h \perp AB$; m and n are the two segments of AB; the chords AC and BC are drawn. What kind of a triangle is ABC? What follows?)

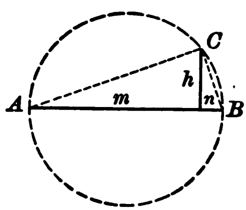

Fig. 163.

6. The diameter of a circle is 13.2 cm. and the perpendicular from a point in the circumference upon the diameter is 4.2 cm.

(a) Find the segments of the diameter.

(b) Find the lengths of the chords from the point in the circumference to the ends of the diameter.

§ **142.** THEOREM IV. ***In any right triangle, the square of the hypotenuse equals the sum of the squares of the two sides.***

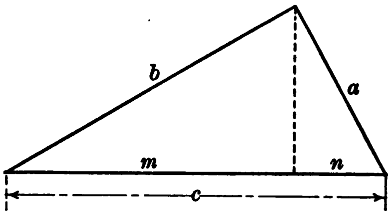

FIG. 164.

In the given right triangle, the altitude is drawn upon the hypotenuse; m and n are the segments of the hypotenuse; a and b are the sides of the right triangle.

1	$\frac{c}{a}=\frac{a}{n}$	Why?
2	$cn=a^2$	1 $\times an$
3	$\frac{c}{b}=\frac{b}{m}$	Why?
4	$cm=b^2$	3 $\times bm$
5	$cn+cm=a^2+b^2$	2 + 4
6	$c(n+m)=a^2+b^2$	5 (Factoring)
7	$c^2=a^2+b^2$	6 (Substitution of c for $n+m$)

State your conclusion.

This theorem is known as the **Pythagorean Theorem.** It is thought that Pythagoras proved this proposition by proportion in a manner similar to the above proof, but of this we are not certain. This theorem can also be proved by means of comparing the areas of similar plane figures constructed on the sides of the right triangle. The theorem is not only the most famous theorem of all geometry; but, as you have seen, it is one of the most useful in the solution of problems.

EXERCISES

1. Builders, using a ball of twine and a 10-ft. pole, apply the Pythagorean Theorem in staking out the foundation of a building in the following way.

A string is stretched between two stakes set at A and B (Fig. 165). This line determines the location of one wall. If there is to be a square corner at B, BC is measured off equal to 8 ft. A 10-ft. pole is placed with its end at C and held in the position shown in the figure. (CD represents the pole.) On another string BN, fastened to stake B, BD is measured off equal to 6 ft. This string is held by a man at N, who brings the string BN so that it just touches the end of the pole at D. A stake is then set at N and the string is fastened to it.

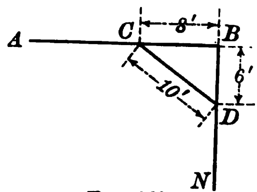

FIG. 165.

Show that the angle B must be a right angle; that is, that the corner is "square" at B.

2. Another method of running one line at right angles to a second line is known as the "rope-stretching" method. To use this method you proceed as follows: Divide a string into segments in the ratio of 3, 4, and 5; join the ends and stretch the string taut, after pins have been put between the segments. Prove that the triangle formed will be a right triangle. If a 50-ft. tape is to be used, what are the segments into which it can be divided conveniently in order to use this method?

3. A square having a side 12.5″ long is inscribed in a circle. Find the diameter of the circle.

4. The diagonal of a rectangle is 45″. If the length of the rectangle is twice its width, find its dimensions.

5. The length of a rectangle exceeds its width by 2″ and the diagonal is 14″. Find the dimensions of the rectangle.

6. The diagonal of a rectangle is 6.1″ and the ratio of its length to its width is 12. Find its dimensions.

7. Find the height of an isosceles triangle whose perimeter is 25 cm. and whose base is half one of the equal sides.

8. Find the diameter of a circle circumscribing a square whose side is $3\sqrt{2}''$.

9. Find the diameter of the round rod from which a square rod 1″ on each side can be cut.

10. Find the diagonal of a cube whose edge is 2″. (SUGGESTION. In Fig. 166, first find *AB*; then, using triangle *ABC*, find *AC*.)

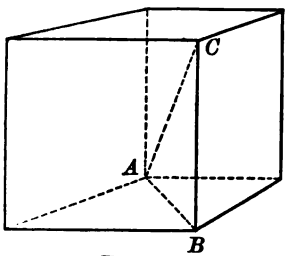

FIG. 166.

11. A kite is inscribed in a circle whose diameter is 6′ (Fig. 167). The length of one of the shorter sides is 3′. Find the length of one of the longer sides, and the perimeter of the kite.

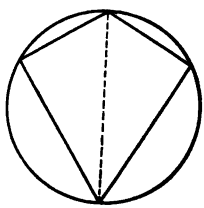
FIG. 167.

12. A certain 60°–30° draftsman's triangle has a hypotenuse of 12″. Find the length of the two sides. (See Ex. 2, page 228.)

13. The bases of an isosceles trapezoid are 30″ and 20″; one of the equal sides is 13″. Find the height of the trapezoid (Fig. 168).

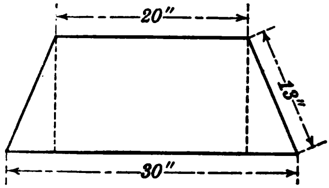

FIG. 168.

CHAPTER XX

MENSURATION

§ **143. Area.** The ***area*** of a surface is the number showing how many square units of a given kind it contains.

§ **144.** THEOREM I. ***The area of a rectangle is equal to the product of its base and height.***

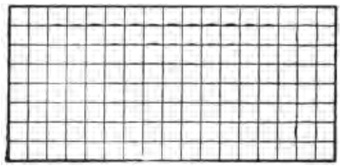

FIG. 169.

The rectangle $ABCD$ contains 136 squares, since there are 8 rows of 17 squares each.

The number of squares thus formed is called the *area* of the rectangle, hence

The area of rectangle $ABCD = 136$ square units.

If we let A stand for the number of square units in the area of a rectangle, and b and h stand for the number of linear units, of the same kind, in the base and height, respectively, then

A (the number of square units) $= b \times h$ square units; that is,

$$A = bh$$

is the *formula* for the area of a rectangle.

NOTE. The square units most commonly used are the following: the square inch, the square foot, the square yard, the square centimeter, and the square meter.

EXERCISES

1. Construct on squared paper a rectangle in which $b=1.2$ in., and $h=0.50$ in. Count the number of unit squares and name the unit. Check your result numerically by substituting the values of b and h in the formula.

2. Construct on squared paper a square on a line 1.4 in. long. Count the number of unit squares and name the unit. Check your result numerically by substituting the values of b and h in the formula for the area of the rectangle.

3. In a square whose side is s, show by a geometric diagram that its area can be expressed by the formula, $A=s\times s$, or s^2.

4. In a rectangle, where $b=\sqrt{3}$ and $h=\sqrt{2}$, we shall assume that the formula for the area still holds true; that is, that $A=\sqrt{3}\times\sqrt{2}=\sqrt{6}$.

Using the following pairs of values for b and h, determine the number value of A in each case. (The method of Approximate Products, pages 13–17, may be used to advantage in these computations.)

b	1.4	1.41	1.414	1.4142	1.41421
h	1.7	1.73	1.732	1.7320	1.73205
A	? (2 figs.)	? (3 figs.)	? (4 figs.)	? (5 figs.)	? (6 figs.)

From an inspection of these products, it seems reasonable to conclude that they constantly approach the exact area of the rectangle; that is, $\sqrt{6}$, which is 2.44949.

5. Find the area of a rectangle with each of the following pairs of dimensions.

b	3.8″	5.6″	$\sqrt{3}$″	$\sqrt{3}$″	7.4 cm.
h	2.6″	0.82″	$\sqrt{7}$″	$\sqrt{6}$″	3.8 cm.
A	?	?	?	?	?

NOTE. Since the base b and the height h of any rectangle may be measured by whole numbers to any required degree of accuracy, if a sufficiently small unit of measure is chosen, it follows that any rectangle may be thought of as consisting of h rows of b square units each; that is, that the area is measured by bh to any required degree of accuracy.

6. Prove that any two rectangles are in the same ratio as the products of their bases and heights.

Let A_1 and A_2 = the areas of the rectangles,
b_1 and b_2 = the bases, and
h_1 and h_2 = the heights, then

$A_1 = b_1h_1$ Why?

$A_2 = b_2h_2$ Why?

$\dfrac{A_1}{A_2} = \dfrac{b_1h_1}{b_2h_2}$ Why?

Ans. $\dfrac{A_1}{A_2} = \dfrac{b_1h_1}{b_2h_2}$.

7. Prove that any two rectangles having equal heights are in the same ratio as their bases.

$\dfrac{A_1}{A_2} = \dfrac{b_1h_1}{b_2h_2}$ (From Ex. 6)

But $h_1 = h_2$, hence what follows?

8. Prove that any two rectangles having equal bases are in the same ratio as their heights.

§ 145. THEOREM II. ***The area of a parallelogram is equal to the product of its base and height.***

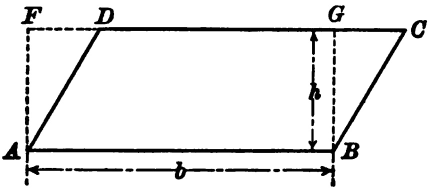

FIG. 170.

In $\square ABCD$, b is the base and h is the height.

Draw $AF \perp CD$ produced.

$ABGF$ is a rectangle.	Why?
$AF = BG$	Why?
$AD = BC$	Why?
rt. $\triangle AFD \cong$ rt. $\triangle BGC$	Why?
$ABCF -$ rt. $\triangle AFD = ABCF -$ rt. $\triangle BGC$	Why?
$\square ABCD =$ rectangle $ABGF$	Why?
$\square ABCD = bh$	Why?

State your conclusion.

EXERCISES

1. Prove that any two parallelograms are in the same ratio as the products of their bases and heights. (SUGGESTION. See Ex. 6, page 264.)

2. Prove that any two parallelograms having equal bases and heights are equal in area.

3. Prove that any two parallelograms having equal bases are in the same ratio as their heights.

4. Prove that any two parallelograms having equal heights are in the same ratio as their bases.

§ **146.** THEOREM III. ***The area of a triangle is equal to one half the product of its base and height.***

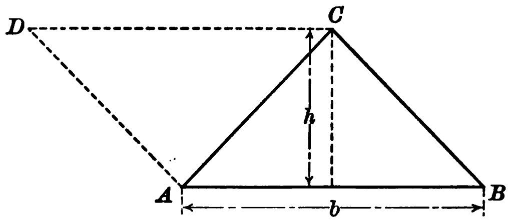

FIG. 171.

In $\triangle ABC$, b is the base and h is the height.

Complete the $\square ABCD$ by drawing $AD \| BC$ and $CD \| AB$

$\square ABCD = bh$ Why?

$\triangle ABC \cong \triangle ACD$ Why?

$\triangle ABC = \frac{1}{2} \square ABCD$ Why?

$\triangle ABC = \frac{1}{2} bh.$ Why?

State your conclusion.

EXERCISES

1. Prove that any two triangles are in the same ratio as the products of their bases and heights.

2. Prove that any two triangles having equal bases and heights are equal in area.

3. Prove that any two triangles having equal bases are in the same ratio as their heights.

4. Prove that any two triangles having equal heights are in the same ratio as their bases.

§ **147**. Theorem IV. *The area of a trapezoid is equal to one half the product of the sum of its bases by its height.*

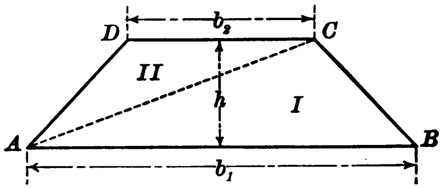

Fig. 172.

In trapezoid $ABCD$, b_1 and b_2 are the bases and h is the height. Draw the diagonal AC.

$$\triangle \text{I} = \tfrac{1}{2}b_1h \qquad \text{Why?}$$

$$\triangle \text{II} = \tfrac{1}{2}b_2h \qquad \text{Why?}$$

$$\triangle \text{I} + \triangle \text{II} = \tfrac{1}{2}b_1h + \tfrac{1}{2}b_2h \qquad \text{Why?}$$

Trapezoid $$ABCD = \tfrac{1}{2}h(b_1 + b_2) \qquad \text{Why?}$$

State your conclusion.

Note. The area of a polygon of 4 or more sides can be found in the following way: draw the longest diagonal of the polygon; then draw perpendiculars to the diagonal from the other vertices of the polygon; find the sum of the areas of all the triangles and trapezoids thus formed. (See Fig. 194, page 279.)

§ **148. Regular Polygons.** A ***regular polygon*** is a polygon that is both equilateral and equiangular.

In a regular polygon the center is equidistant from the vertices and equidistant from the sides.

The distance from the center to one side is often called the *short radius;* the distance from the center to one vertex is often called the *long radius.*

The ***perimeter*** of a polygon is the sum of all its sides.

§ **149**. THEOREM V. *The area of a regular polygon is equal to one half the product of its perimeter and the short radius.*

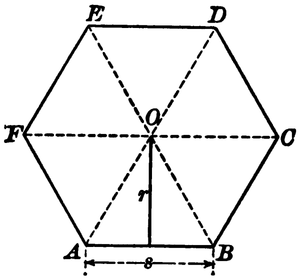

FIG. 173.

In the regular polygon $ABCDEF$, draw the long radii AO, BO, CO, DO, EO, and FO.

This gives as many triangles as the polygon has sides.

Draw a short radius.

Then, $AO=BO=CO=DO=EO=FO$. Why?

Also, $\angle AOB = \angle BOC = \angle COD = \angle DOE$, etc. Why?

Represent the perimeter by p, the side by s, and the short radius by r.

The triangles AOB, BOC, COD, etc., are congruent isosceles triangles. Why?

$$\triangle AOB = \tfrac{1}{2}rs \qquad \text{Why?}$$

$$\triangle BOC = \tfrac{1}{2}rs, \text{ etc.} \qquad \text{Why?}$$

$\triangle AOB + \triangle BOC + \triangle COD + \text{etc.} = \tfrac{1}{2}rs + \tfrac{1}{2}rs + \tfrac{1}{2}rs + \text{etc.}$

But polygon $ABCDEF$ is equal to the sum of all the triangles.

Hence, polygon $ABCDEF = \tfrac{1}{2}r(s+s+s+\text{etc.})$ or

Polygon $ABCDEF = \tfrac{1}{2}rp$. Why?

State your conclusion.

§ 150. THEOREM VI. ***In any right triangle, the square on the hypotenuse is equal to the sum of the squares on the two sides.***

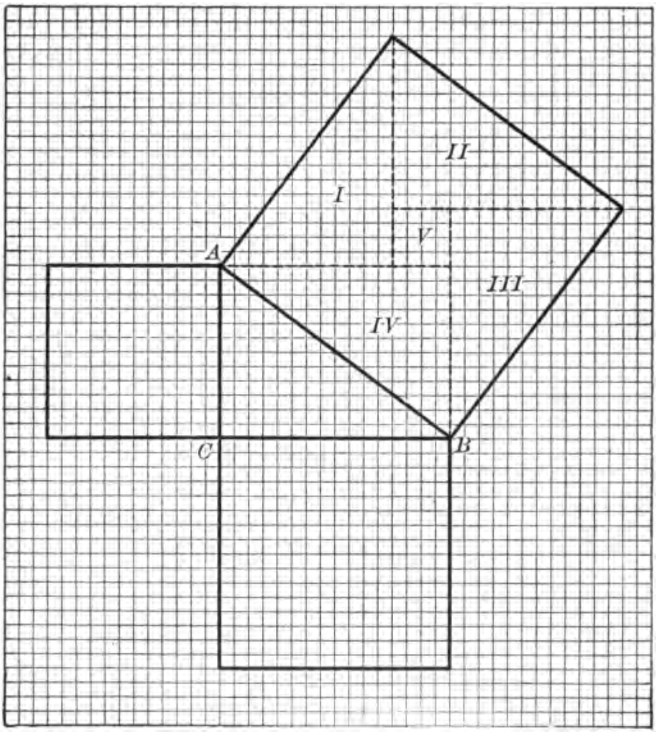

FIG. 174.

The square on AC+the square on BC = 144+256

= 400 (square units)

$\triangle \text{I} + \triangle \text{II} + \triangle \text{III} + \triangle \text{IV} + \triangle \text{V} = 96 + 96 + 96 + 96 + 16$

= 400 (square units)

State your conclusion.

NOTE. The Pythagorean Theorem was proved on page 258 by means of proportion; that form of proof is called an *algebraic* proof. The method of proof used here, — that is, by a comparison of the areas of plane figures, constructed on the sides of the right triangle, — is called a *geometric* proof. There are many other ways of proving this theorem by the geometric method, but the method used here is probably one of the simplest.

EXERCISES

1. Prove that the square on one side of a right triangle is equal to the square on the hypotenuse minus the square on the other side.

2. Construct a square equal to the sum of two given squares. Carry out the construction as shown in Fig. 175, using ruler and compasses.

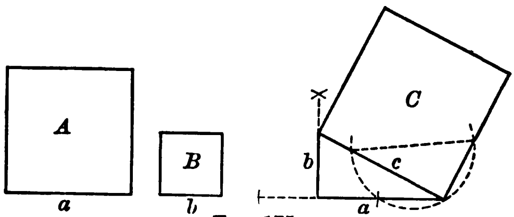

FIG. 175.

(SUGGESTION. On a given line lay off a segment a; erect a perpendicular at one end of a and on this perpendicular lay off b; join the other ends of the segments a and b.)

Prove that the square on c is the required square.

3. Construct a square equal to the difference of two given squares.

(SUGGESTION. Reverse the construction in Ex. 2 as follows: after erecting a perpendicular at one end of a, use the other end of a as a center, and with a radius c describe an arc cutting the perpendicular.)

§ **151.** Theorem VII. ***The ratio of the circumference of a circle to its diameter is a constant.***

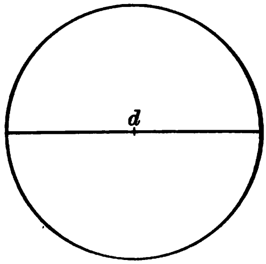

Fig. 176.

Using a tape measure, find the distance around a circular disk to the nearest tenth of an inch; also, measure its diameter to the same degree of accuracy.

Divide the number of inches in the circumference by the number of inches in the diameter. What is the ratio?

In a similar way, measure the circumferences and diameters of several circular objects; such as, an ash can, a pail, a half dollar, a bicycle wheel, etc. Find the ratio of the circumference to the diameter in each case.

State your conclusion.

Note. If your measurements were very accurate, you would find the ratio to be nearly 3.1416. This mixed decimal is commonly expressed by the symbol π (read *pi*).

EXERCISES

1. Prove that the circumference of a circle is equal to π times its diameter.

2. Prove that the circumference of a circle is equal to 2π times its radius.

§ **152.** THEOREM VIII. ***The area of a circle is equal to one half its circumference times its radius.***

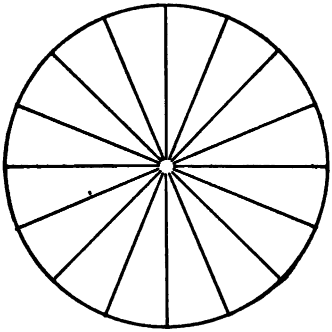

FIG. 177.

Cut the surface inclosed by a circle into any number of equal parts, say 16, as shown in Fig. 177. Fit them together as in Fig. 178.

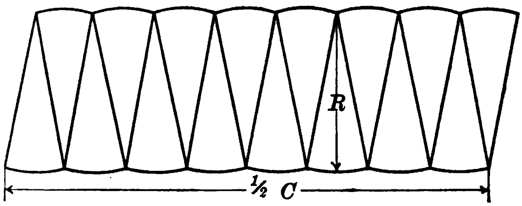

FIG. 178.

Figure 178 resembles a parallelogram, having for its base one half the circumference of the circle, and for its height the radius of the circle.

For the parallelogram, we have

$$A = bh \text{ square units.}$$

Hence, for the circle, we have

$$A = \tfrac{1}{2}cr \text{ square units.}$$

EXERCISES

1. Letting c = the circumference of a circle and r = the radius, write a formula for the area, A.

2. Prove that the area of a circle may be expressed by the formula: $A = \pi r^2$.

3. In Fig. 179, count the number of small squares and parts of squares in one quarter of the circle; multiply the total number of whole squares that you get by 4. Compare your result with the statement in Ex. 2.

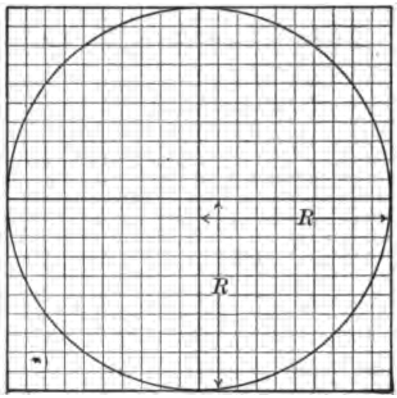

FIG. 179.

4. Prove that the area of a circle may be expressed by the formula: $A = \frac{1}{4}\pi d^2$.

5. Prove that the areas of two circles are in the same ratio as the squares of their radii; as the squares of their diameters.

6. In Fig. 180, the shaded part is a *sector* of a circle. It is the same fractional part of the area of the circle as its angle is of 360°. How would you find the area of a sector?

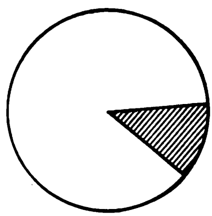
FIG. 180.

CHAPTER XXI

FORMULAS OF MENSURATION

§ 153. Formulas for the Mensuration of Plane Figures. In formulas (1) to (5), A = area; p = perimeter; b, b_1, b_2, = bases; h = height; and s = side.

(1)	*Rectangle:*	$A=bh$	$p=2b+2h$
(2)	*Square:*	$A=s^2$	$p=4s$
(3)	*Parallelogram:*	$A=bh$	
(4)	*Triangle:*	$A=\frac{1}{2}bh$	
(5)	*Trapezoid:*	$A=\frac{1}{2}h(b_1+b_2)$	

In formulas (6) and (7), A = area, c = circumference, r = radius, d = diameter, and $\pi=3.1416$ (3.14 to three figures).

(6) *Circle:* $c=\pi d$, or $c=2\pi r$

(7) *Circle:* $A=\pi r^2$, or $A=\frac{1}{4}\pi d^2$

(8) *Angles of a plane triangle:*

$$\angle A+\angle B+\angle C=180^\circ,$$

where A, B, and C = the angles of the triangle.

(9) *Sides of a right triangle:*

$$c^2=a^2+b^2,$$

where c = hypotenuse, and a and b = the sides.

§ 154. Solids. A ***rectangular block*** is a solid bounded by six rectangles, called *faces* (Fig. 181).

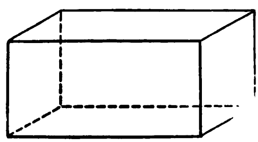

FIG. 181.

A ***cube*** is a rectangular block all of whose faces are squares (Fig. 182).

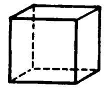

FIG. 182.

A ***right prism*** is a solid bounded by two congruent polygons, called *bases*, and by rectangles, called *lateral faces*. The *height*, or *altitude*, is the perpendicular distance between the bases (Fig. 183).

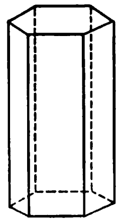

FIG. 183.

A ***regular pyramid*** is a solid bounded by a regular polygon, called the *base*, and congruent isosceles triangles, called *lateral faces*, meeting at a common point called the *vertex* (Fig. 184).

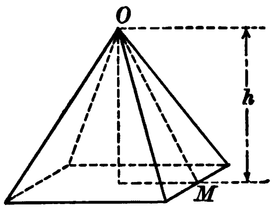

FIG. 184.

The *height*, or *altitude*, is the perpendicular distance from the vertex to the base (h in Fig. 184). The ***slant height*** is the height of one of the lateral faces (OM in Fig. 184).

A ***frustum of a regular pyramid*** is the solid included between its base and a section made by a plane parallel to the base.

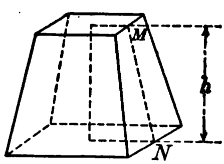

FIG. 185.

The *height*, or *altitude*, is the perpendicular distance between the bases (h in Fig. 185). The *slant height* is the height of one of the lateral faces (MN in Fig. 185). The lateral faces are congruent isosceles trapezoids.

A ***right circular cylinder*** is the solid formed when a *rectangle* is turned through a complete revolution about one of its sides as an axis.

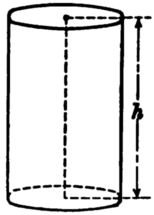

FIG. 186.

The *height*, or *altitude*, is the perpendicular distance between the bases (h in Fig. 186).

A ***right circular cone*** is the solid formed when a right triangle is turned through a complete revolution about one of its perpendicular sides as an axis.

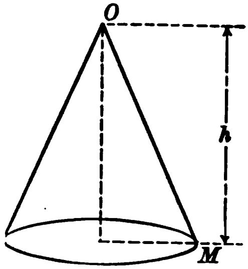

FIG. 187.

The *height*, or *altitude*, is the perpendicular distance from the vertex to the base (h in Fig. 187). The *slant height* is the distance from the vertex to any point in the circumference of the base (OM in Fig. 187).

A ***frustum of a right circular cone*** is the portion of a right circular cone included between the base and a section made by a plane parallel to the base.

The base of the cone and the parallel section are together called the *bases* of the frustum.

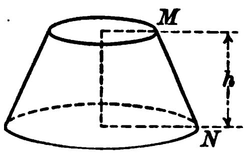

FIG. 188.

The *height*, or *altitude*, of the frustum is the perpendicular distance between the bases (h in Fig. 188). The *slant height* is the distance from a point in the circumference of the upper base to the corresponding point in the circumference of the lower base (MN in Fig. 188).

A *sphere* is a round solid inclosed by a surface, all points of which are equidistant from a point within called the center (Fig. 189).

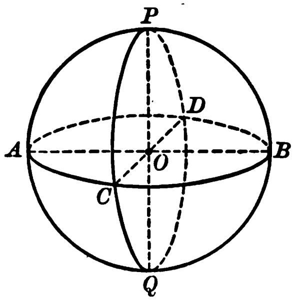

Fig. 189.

§ 155. Formulas for the Mensuration of Solids. The *volume* of a solid is the number showing how many *cubic units* of a given kind it contains.

In Fig. 190, the rectangular block consists of 4 layers, each made up of 15 cubic units, each equal to M.

Hence, the volume of the block $=60$ cubic units.

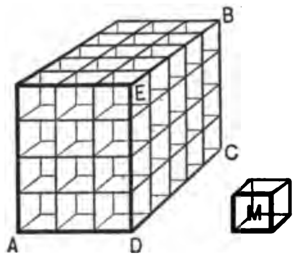

Fig. 190.

In formulas (1) to (5), $V=$ volume; $S=$ area of lateral surface; B, B_1, $B_2=$ areas of bases; p, p_1, $p_2=$ perimeters of bases; $h=$ height; $l=$ length (1), or slant height (4) and (5); and $e=$ edge.

(1) *Rectangular block:* $V=lwh$

(2) *Cube:* $V=e^3$

(3) *Right prism:* $V=Bh$ $\qquad S=ph$

(4) *Regular pyramid:* $V=\frac{1}{3}Bh$ $\qquad S=\frac{1}{2}pl$

(5) *Frustum of regular pyramid:* $V=\frac{1}{3}h(B_1+B_2+\sqrt{B_1B_2})$

$S=\frac{1}{2}(p_1+p_2)l$

In formulas (6) to (9), V = volume; S = area of curved surface; r, r_1, r_2 = radii; h = height; l = slant height; and $\pi = 3.1416$ (3.14 to three figures).

(6) *Right circular cylinder:*	$V = \pi r^2 h$	$S = 2\pi r h$
(7) *Right circular cone:*	$V = \frac{1}{3}\pi r^2 h$	$S = \pi r l$
(8) *Frustum of right circular cone:*	$V = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$	$S = \pi(r_1 + r_2)l$
(9) *Sphere:*	$V = \frac{4}{3}\pi r^3$	$S = 4\pi r^2$

PROBLEMS

1. On a map drawn to a scale of 60 mi. to the inch, what area is represented by a strip 3″ long and 2″ wide?

2. Find the cross-sectional area of an excavation for a railroad bed, if it is 6′ wide at the top, 5′ wide at the bottom, and 1.5′ deep. How many cubic yards of earth are removed in a section of 100 yards? (See Fig. 191.)

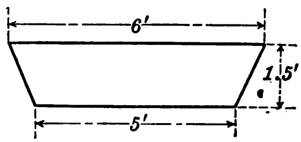

Fig. 191.

3. Find the area of a right triangle if its hypotenuse is 15 cm. and one side is 9 cm.

4. Find the area of a right triangle if its hypotenuse is $5\sqrt{2}''$ and one side is 5″.

5. Find the area of an *isosceles* right triangle whose hypotenuse is $10\sqrt{2}''$.

6. Find the cross-sectional area of the Z-bar, with dimensions as shown in Fig. 192.

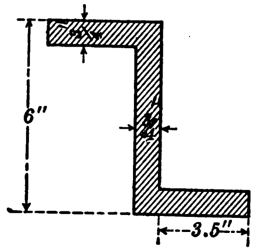

Fig. 192.

7. Find the cross-sectional area of the channel-iron, with dimensions as shown in Fig. 193.

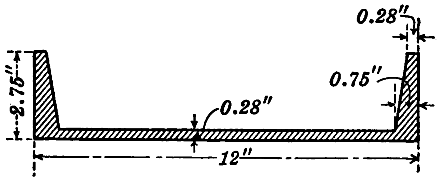

Fig. 193.

8. Figure 194 is the plan of a field. Make a plan to the scale of 50′ to 1″; draw the diagonal and draw the altitudes of the triangles as shown in the figure. Measure the lines necessary for finding the areas of the triangles ABC and ADC. Find the approximate number of square feet in the area of the field.

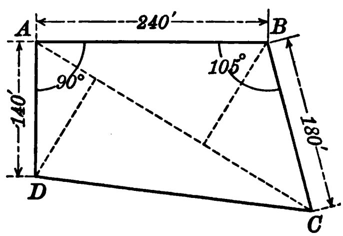

Fig. 194.

9. The perimeters of an equilateral triangle, of a square, of a regular hexagon, and of a circle are each equal to 60 cm. Find the area of each and arrange them in order of size.

10. How many revolutions does a 40″ automobile wheel make in going one mile?

11. Four circular boiler plates 24″ in diameter are cut from a plate 48″ by 48″.

(*a*) Find the number of square inches of stock wasted.

(*b*) Find the per cent of stock wasted.

12. Find the area of a sector of a 10-cm. circle if the angle of the sector is 30°. (See Ex. 6, page 273.)

13. In Fig. 195, the diameter of the circle is 12″ and angle $AOB = 60°$.

(*a*) Find the length of the arc AB.

(*b*) Find the area of the sector AOB.

(*c*) Find the area of the triangle AOB.

(*d*) Find the area inclosed by the chord AB and its arc.

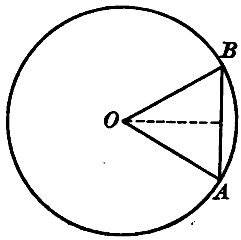

FIG. 195.

14. The diameter of a piston of an engine is 22″. The pressure on the end of the piston is 100 lb. per square inch. Find the total pressure on the end of the piston.

15. The inside diameter of a certain water pipe is 10″. The pipe is 1″ thick. Find the cross-sectional area of the metal in the pipe (Fig. 196).

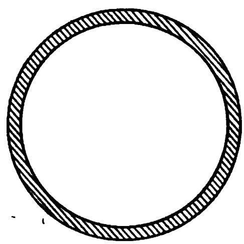

FIG. 196.

16. Find the weight of 4-ft. length of the pipe in Ex. 15. The density of the metal is 0.26 lb. per cubic inch.

(SUGGESTION. To get the volume, multiply the cross-sectional area of the metal by the length of the pipe.)

17. Neglecting overlapping, how many square feet of tin are required to make a stovepipe 8′ long and 6″ in diameter?

18. Find the total area (curved surface and two ends) of a tin can, 4″ in diameter and 6″ high.

19. How many ends for cans of the dimensions given in Ex. 18 can be cut from a sheet of tin 40″ by 48″? How many square inches are wasted? What per cent is wasted?

20. A plate glass window is 8′ by 10′ and $\frac{1}{4}$″ thick. Find its weight, the density of the glass being 172 lb. per cubic foot.

21. How many square feet of lead will be required to cover the bottom and sides of a rectangular tank, 3.5 ft. deep, 4.2 ft. long, and 3.5 ft. wide? What will be the weight of the lead if it is 0.05 in. thick and a cubic inch weighs 0.41 lb.?

22. The Great Pyramid in Egypt is about 450 ft. high and its base is a square about 764 ft. on each side. Find the approximate number of cubic yards in its volume.

23. Figure 197 is the plan of a pavement. What volume of cement $\frac{1}{4}$″ thick is required to cover it?

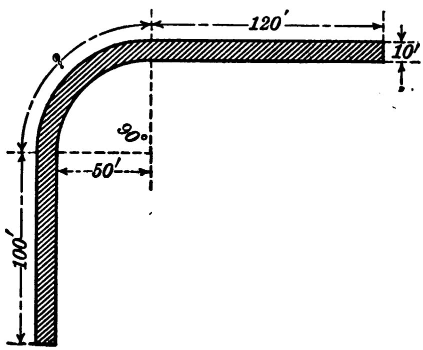

FIG. 197.

24. A cylindrical pail is 45 cm. deep and 15 cm. in diameter (inside). Find its contents in cubic centimeters. How many liters will it hold?

25. A cast-iron bar is 2.8 cm. in diameter and 2.5 m. long.

(*a*) Find its volume in cubic centimeters.

(*b*) A piece 12 mm. long is cut off. What per cent of the bar is left?

26. Find the weight of a certain steel cone 6 in. in diameter and 5 in. high. A cubic inch of the steel weighs 0.28 lb.

27. Find the weight in kilograms of a certain iron ball 11 cm. in diameter. The density of the iron is 7.2 grams per cubic centimeter.

28. Figure 198 is the plan of a hemispherical head for a steel bolt whose diameter is $1\frac{1}{4}''$. Find the weight of the head, if a cubic inch of the steel weighs 0.28 lb.

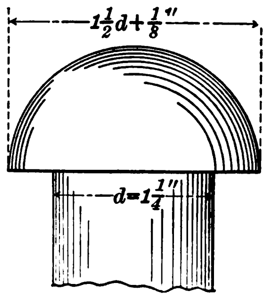

FIG. 198.

29. The area of a circle in square inches and its circumference in inches are expressed by the same number. Find its diameter.

30. The volume of a cylinder in cubic centimeters and the area of its curved surface in square centimeters are expressed by the same number. Find its diameter.

31. Cylindrical cans of a certain brand of condensed milk are shipped to France in boxes containing 2 layers of 12 cans each. Before the boxes are nailed up, the

spaces between the cans are filled with wheat. The diameter of a can is 3″. The box is 12″ long, 9″ wide, $6\frac{1}{2}''$ deep (inside dimensions). If one bushel of wheat occupies approximately $\frac{5}{4}$ cu. ft., how many bushels of wheat can be sent in 10,000 such boxes?

32. In building a retaining wall of concrete the proportion of gravel, sand, and cement is 5 to 3 to 1. The wall is 50 ft. long, 4 ft. high, and 10 in. thick. How many cubic feet of each kind of material is needed, if there is an allowance of 10% for waste?

33. Find the weight per linear foot of lead pipe of $1\frac{1}{2}''$ bore (inside diameter) and $\frac{3}{16}''$ thick. A cubic inch of the lead weighs 0.41 lb.

34. A cylindrical pail is 20 centimeters in diameter (inside). How deep must it be to hold 5 liters?

35. A cubic inch of copper is drawn into a wire 5 mm. in diameter. Find the length of the wire in meters.

36. A disk 10 cm. in diameter and 0.8 cm. thick is turned out of a piece of stock 12 cm. square and 1 cm. thick. How much is wasted in shavings? What per cent is wasted?

37. A conduit is made of concrete (shaded part in Fig. 199).

(*a*) Find the total cross-sectional area of the entire figure.

(*b*) Find the total cross-sectional area of the shaded part.

(*c*) Find the number of cubic feet of concrete needed for 400 yards of this conduit.

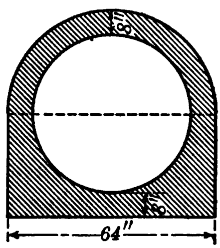

Fig. 199.

38. A galvanized iron pail, in the shape of a frustum of a cone, has the dimensions given in Fig. 200.

(*a*) Find the length of AB.

(*b*) Find the number of square inches in the bottom.

(*c*) Find the number of square inches in the conical part.

(*d*) Find the number of cubic inches in the contents.

(*e*) How many gallons will the pail hold?

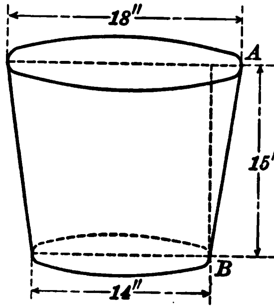

FIG. 200.

39. Figure 201 represents a certain type of milk can.

(*a*) Find the area of the bottom.

(*b*) Find the area of the two cylindrical surfaces.

(*c*) Find the length of AB.

(*d*) Find the area of the surface of the frustum.

(*e*) Find the total surface of the entire can (without cover).

(*f*) Find the total number of square inches of tin sheeting needed to make the can, allowing 10% for the seams.

(*g*) Find the volume of the larger cylindrical part.

(*h*) Find the volume of the frustum.

(*i*) How many quarts will the can hold, to within a tenth of a quart?

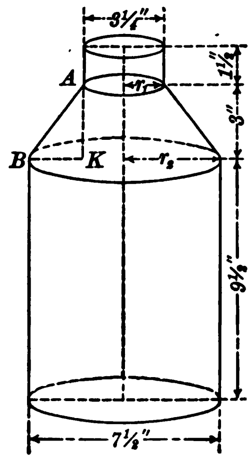

FIG. 201.

APPENDIX

TABLES OF SQUARES AND SQUARE ROOTS

§ 156. **Squares of Numbers.** The table on pages 288–290 gives the squares of all three-figure numbers from 1.00 to 10.00. The first *two* digits of the number are in the *column n*, and the third digit is in the *row* beginning with *n*. The *square* is at the intersection of the row and column on which the number is located. For example, $4.76^2 = 22.66$.

To find the square of a number less than 1 or more than 10, it is necessary *to make the number fit the table.*

$35^2 = (3.5 \times 10)^2 = 12.25 \times 100 = 1225$; hence, when you divide the given number by 10 to make it fit the table, the square of the given number is 100 (10^2) times the square found in the table.

$0.35^2 = \left(\frac{3.5}{10}\right)^2 = \frac{12.25}{100} = 0.1225$; hence, when you multiply the given number by 10 to make it fit the table, the square of the given number is $\frac{1}{100}$ $\left(\frac{1}{10^2}\right)$ times the square found in the table.

EXAMPLE 1. Find the square of 48.6.

SOLUTION. (*a*) Making 48.6 fit the table, $\frac{48.6}{10} = 4.86$.

(*b*) Finding the square of 4.86 in the table, $4.86^2 = 23.62$.

(*c*) Making the square 23.62 fit the number 48.6,

$$48.6^2 = (4.86 \times 10)^2 = 23.62 \times 100 = 2362$$

Ans. **2362.**

EXAMPLE 2. Find the square of 0.346.

SOLUTION. (*a*) Making 0.346 fit the table,

$$0.346=\frac{3.46}{10}$$

(*b*) Finding the square of 3.46 in the table, $3.46^2=11.97$.

(*c*) Making the square 11.97 fit the number 0.346,

$$0.346^2=\left(\frac{3.46}{10}\right)^2=\frac{11.97}{100}=0.1197$$

Ans. 0.1197.

§ 157. The Square Roots of Numbers. The table on pages 288–290 gives the square roots of numbers from 1 to 100. To find the square root of a number, look in the table for the square nearest the given number. This square will be at the intersection of a row and a column. The first *two* digits of the *square root* are in *column n* in that row in which the square is located and the *third* digit is in the row beginning with *n* in that column in which the square is located. For example, the square root of 11.16 is 3.34; the square root of 14 is 3.74.

To find the square root of a number less than 1 or more than 100, it is necessary to make the number fit the table.

$\left(\frac{n}{10}\right)^2=\frac{n^2}{100}$; hence, in order to make a square larger than 100 fit the table, it is necessary to divide the square by 100 (or 100^2, etc). Then the required square root is 10 (or 10^2, etc.) times the number found in the table.

$(10n)^2=100n^2$; hence, in order to make a square less than 1 fit the table, it is necessary to multiply the square by 100 (or 100^2, etc.). Then the required square root is $\frac{1}{10}$ (or $\frac{1}{10^2}$, etc.) times the number found in the table.

EXAMPLE 1. Find the square root of 392.

SOLUTION. (*a*) Making the square 392 fit the table,

$$\frac{392}{100}=3.92$$

(*b*) Finding the square root of 3.92 in the table,

$$\sqrt{3.92}=1.98$$

(*c*) Making the square root 1.98 fit the square 392,

$$\sqrt{392}=\sqrt{100\times3.92}=10\times1.98=19.8 \qquad \textit{Ans.}\ 19.8.$$

EXAMPLE 2. Find the square root of 3920.

SOLUTION. (*a*) Making the square 3920 fit the table,

$$\frac{3920}{100}=39.2$$

(*b*) Finding the square root of 39.2 in the table,

$$\sqrt{39.2}=6.26$$

(*c*) Making the square root 6.26 fit the square 3920,

$$\sqrt{3920}=\sqrt{100\times39.2}=10\times6.26=62.6 \qquad \textit{Ans.}\ 62.6.$$

EXAMPLE 3. Find the square root of 0.455.

SOLUTION. (*a*) Making the square 0.455 fit the table,

$$0.455\times100=45.5$$

(*b*) Finding the square root of 45.5 in the table,

$$\sqrt{45.5}=6.75$$

(*c*) Making the square root 6.75 fit the square 0.455,

$$\sqrt{0.455}=0.675 \qquad \textit{Ans.}\ 0.675.$$

EXAMPLE 4. Find the square root of 0.0455.

SOLUTION. (*a*) Making the square 0.0455 fit the table,

$$0.0455\times100=4.55$$

(*b*) Finding the square root of 4.55 in the table,

$$\sqrt{4.55}=2.13$$

(*c*) Making the square root 2.13 fit the square 0.0455,

$$\sqrt{0.0455}=0.213. \qquad \textit{Ans.}\ 0.213.$$

§ 158. TABLE I—SQUARES OF NUMBERS FROM 1.00 TO 10.00

n	0	1	2	3	4	5	6	7	8	9
1.0	1.000	1.020	1.040	1.061	1.082	1.102	1.124	1.145	1.166	1.188
1.1	1.210	1.232	1.254	1.277	1.300	1.322	1.346	1.369	1.392	1.416
1.2	1.440	1.464	1.488	1.513	1.538	1.562	1.588	1.613	1.638	1.664
1.3	1.690	1.716	1.742	1.769	1.796	1.822	1.850	1.877	1.904	1.932
1.4	1.960	1.988	2.016	2.045	2.074	2.102	2.132	2.161	2.190	2.220
1.5	2.250	2.280	2.310	2.341	2.372	2.402	2.434	2.465	2.496	2.528
1.6	2.560	2.592	2.624	2.657	2.690	2.722	2.756	2.789	2.822	2.856
1.7	2.890	2.924	2.958	2.993	3.028	3.062	3.098	3.133	3.168	3.204
1.8	3.240	3.276	3.312	3.349	3.386	3.422	3.460	3.497	3.534	3.572
1.9	3.610	3.648	3.686	3.725	3.764	3.802	3.842	3.881	3.920	3.960
2.0	4.000	4.040	4.080	4.121	4.162	4.202	4.244	4.285	4.326	4.368
2.1	4.410	4.452	4.494	4.537	4.580	4.622	4.666	4.709	4.752	4.796
2.2	4.840	4.884	4.928	4.973	5.018	5.062	5.108	5.153	5.198	5.244
2.3	5.290	5.336	5.382	5.429	5.476	5.522	5.570	5.617	5.664	5.712
2.4	5.760	5.808	5.856	5.905	5.954	6.002	6.052	6.101	6.150	6.200
2.5	6.250	6.300	6.350	6.401	6.452	6.502	6.554	6.605	6.656	6.708
2.6	6.760	6.812	6.864	6.917	6.970	7.022	7.076	7.129	7.182	7.236
2.7	7.290	7.344	7.398	7.453	7.508	7.562	7.618	7.673	7.728	7.784
2.8	7.840	7.896	7.952	8.009	8.066	8.122	8.180	8.237	8.294	8.352
2.9	8.410	8.468	8.526	8.585	8.644	8.702	8.762	8.821	8.880	8.940
3.0	9.000	9.060	9.120	9.181	9.242	9.302	9.364	9.425	9.486	9.548
3.1	9.610	9.672	9.734	9.797	9.860	9.922	9.986	10.05	10.11	10.18
3.2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82
3.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49
3.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89
3.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62
3.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36
3.8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	14.98	15.05	15.13
3.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92
4.0	16.00	16.08	16.16	16.24	16.32	16.40	14.48	16.56	14.65	16.73
4.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56
4.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40
4.3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27
4.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07
4.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00
4.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94
4.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91
4.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90

n	0	1	2	3	4	5	6	7	8	9
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91
5.1	26.01	26.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94
5.2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98
5.3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05
5.4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25
5.6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38
5.7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52
5.8	33.64	33.76	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69
5.9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09
6.1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32
6.2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56
6.3	39.69	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83
6.4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12
6.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43
6.6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76
6.7	44.89	45.02	45.16	45.29	45.43	45.56	45.70	45.83	45.97	46.10
6.8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47
6.9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27
7.1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70
7.2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14
7.3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61
7.4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10
7.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61
7.6	57.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	58.98	59.14
7.7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68
7.8	60.84	60.99	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25
7.9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45
8.1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08
8.2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72
8.3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39
8.4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08
8.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79
8.6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52
8.7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.09	77.26
8.8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03
8.9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82

n	0	1	2	3	4	5	6	7	8	9
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63
9.1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46
9.2	84.64	84.82	85.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30
9.3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17
9.4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06
9.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97
9.6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90
9.7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84
9.8	96.04	96.24	96.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81
9.9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80

TABLE II — RECIPROCALS OF NUMBERS FROM 1 TO 9.9

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	1.000	0.909	0.833	0.769	0.714	0.667	0.625	0.588	0.556	0.526
2	0.500	0.476	0.455	0.435	0.417	0.400	0.385	0.370	0.357	0.345
3	0.333	0.323	0.313	0.303	0.294	0.286	0.278	0.270	0.263	0.256
4	0.250	0.244	0.238	0.233	0.227	0.222	0.217	0.213	0.208	0.204
5	0.200	0.196	0.192	0.189	0.185	0.182	0.179	0.175	0.172	0.169
6	0.167	0.164	0.161	0.159	0.156	0.154	0.152	0.149	0.147	0.145
7	0.143	0.141	0.139	0.137	0.135	0.133	0.132	0.130	0.128	0.127
8	0.125	0.123	0.122	0.120	0.119	0.118	0.116	0.115	0.114	0.112
9	0.111	0.110	0.109	0.108	0.106	0.105	0.104	0.103	0.102	0.101

TABLE III — TRIGONOMETRIC RATIOS

[The abbreviation *hyp* means *hypotenuse; adj* means the *side adjacent* to the angle; *opp* means the *side opposite* the angle.

Angle	Sine (opp/hyp)	Cosine (adj/hyp)	Tangent (opp/adj)	Angle	Sine (opp/hyp)	Cosine (adj/hyp)	Tangent (opp/adj)
0°	.000	1.000	.000	**45°**	.707	.707	1.000
1°	.017	1.000	.017	46°	.719	.695	1.036
2°	.035	.999	.035	47°	.731	.682	1.072
3°	.052	.999	.052	48°	.743	.669	1.111
4°	.070	.998	.070	49°	.755	.656	1.150
5°	.087	.996	.087	**50°**	.766	.643	1.192
6°	.105	.995	.105	51°	.777	.629	1.235
7°	.122	.993	.123	52°	.788	.616	1.280
8°	.139	.990	.141	53°	.799	.602	1.327
9°	.156	.988	.158	54°	.809	.588	1.376
10°	.174	.985	.176	**55°**	.819	.574	1.428
11°	.191	.982	.194	56°	.829	.559	1.483
12°	.208	.978	.213	57°	.839	.545	1.540
13°	.225	.974	.231	58°	.848	.530	1.600
14°	.242	.970	.249	59°	.857	.515	1.664
15°	.259	.966	.268	**60°**	.866	.500	1.732
16°	.276	.961	.287	61°	.875	.485	1.804
17°	.292	.956	.306	62°	.883	.469	1.881
18°	.309	.951	.325	63°	.891	.454	1.963
19°	.326	.946	.344	64°	.899	.438	2.050
20°	.342	.940	.364	**65°**	.906	.423	2.145
21°	.358	.934	.384	66°	.914	.407	2.246
22°	.375	.927	.404	67°	.921	.391	2.356
23°	.391	.921	.424	68°	.927	.375	2.475
24°	.407	.914	.445	69°	.934	.358	2.605
25°	.423	.906	.466	**70°**	.940	.342	2.747
26°	.438	.899	.488	71°	.946	.326	2.904
27°	.454	.891	.510	72°	.951	.309	3.078
28°	.469	.883	.532	73°	.956	.292	3.271
29°	.485	.875	.554	74°	.961	.276	3.487
30°	.500	.866	.577	**75°**	.966	.259	3.732
31°	.515	.857	.601	76°	.970	.242	4.011
32°	.530	.848	.625	77°	.974	.225	4.331
33°	.545	.839	.649	78°	.978	.208	4.705
34°	.559	.829	.675	79°	.982	.191	5.145
35°	.574	.819	.700	**80°**	.985	.174	5.671
36°	.588	.809	.727	81°	.988	.156	6.314
37°	.602	.799	.754	82°	.990	.139	7.115
38°	.616	.788	.781	83°	.993	.122	8.144
39°	.629	.777	.810	84°	.995	.105	9.514
40°	.643	.766	.839	**85°**	.996	.087	11.430
41°	.656	.755	.869	86°	.998	.070	14.301
42°	.669	.743	.900	87°	.999	.052	19.081
43°	.682	.731	.933	88°	.999	.035	28.636
44°	.695	.719	.966	89°	1.000	.017	57.290
45°	.707	.707	1.000	**90°**	1.000	.000	——

TABLE IV—IMPORTANT NUMBERS

A. *Units of Length*

English Units		Metric Units	
12 inches (in.)	= 1 foot (ft.)	10 millimeters (mm.)	= 1 centimeter (cm.)
3 feet	= 1 yard (yd.)	10 centimeters	= 1 decimeter (dm.)
$5\frac{1}{2}$ yards	= 1 rod (rd.)	10 decimeters	= 1 meter (m.)
320 rods	= 1 mile (mi.)	10 meters	= 1 dekameter (Dm.)
		1000 meters	= 1 kilometer (Km.)

English to Metric	Metric to English
1 in. = 2.5400 cm.	1 cm. = 0.3937 in.
1 ft. = 30.480 cm.	1 m. = 39.37 in. = 3.2808 ft.
1 mi. = 1.6093 Km.	1 Km. = 0.6214 mi.

B. *Units of Area or Surface*

1 square yard = 9 square feet = 1296 square inches
1 acre (A.) = 160 square rods = 4840 square yards
1 square mile = 640 acres = 102400 square rods

C. *Units of Measurement of Capacity*

Dry Measure		Liquid Measure	
2 pints (pt.)	= 1 quart (qt.)	4 gills (gi.)	= 1 pint (pt.)
8 quarts	= 1 peck (pk.)	2 pints	= 1 quart (qt.)
4 pecks	= 1 bushel (bu.)	4 quarts	= 1 gallon (gal.)
		1 gallon	= 231 cu. in.

D. *Metric Units to English Units*

1 liter = 1000 cu. cm. = 61.02 cu. in. = 1.0567 liquid quarts
1 quart = .94636 liter = 946.36 cu. cm.
1000 grams = 1 kilogram (Kg.) = 2.2046 pounds (lb.)
1 pound = .453593 kilogram = 453.59 grams

E. *Other Numbers*

π = ratio of circumference to diameter of a circle
= 3.14159265
1 radian = angle subtended by an arc equal to the radius
$= 57^\circ\, 17'\, 44''.8 = 57^\circ.2957795 = 180^\circ/\pi$
1 degree = 0.01745329 radian, or $\pi/180$ radians
Weight of 1 cu. ft. of water = 62.425 lb.

INDEX

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THE following pages contain advertisements of a few of the Macmillan books on kindred subjects